ON AN OPTIMAL CONTROL PROBLEM

(in Connection with the Theory of Orientor Fields of A. Marchaud and S. K. Zaremba)

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Solutions of ordinary differential systems are trajectories of corresponding vector fields. It may happen that the right hand sides of the system are approximately known up to a given accuracy. If this is the case then we have to deal rather with the more general theory of differential inequalities, which is closely related (see [3]) to a theory developed independently by A. Marchaud [8], [9] and S. K. Zaremba [26], [27]. The last one we will call the theory of orientor fields in contrast to the theory of vector fields. Both authors have used the notion of contingens (or paratingens) in the sense of G. Bouligand [5] in order to define a trajectory of an orientor field. They have established certain properties $\mathscr{P}_0 - \mathscr{P}_3$ of these trajectories known before for solutions of ordinary differential systems. Those properties are: compactness, Kneser's and Hukuhara's property. As an immediate consequence of these facts one can get optimal properties $\mathscr{P}_4$ and $\mathscr{P}_5$.

I learned the results of Zaremba's dissertation before the second world war, since I was a referee of that paper. Then a few years ago I came across with some results on optimal control and I have noticed a close connection between the optimal control problem and the theory of Marchaud-Zaremba. This connection is seen clearly in the following way. We eliminate the control u from the control system $S(f, C)$. This elimination leads to the definition of control counter-domain $N(t, x)$. The last is an orientor field "associated" with the system $S(f, C)$. We introduce also the convex hull $E(t, x)$ of $N(t, x)$ and a further field $Q(t, x)$ = tendor $N(t, x)$ (a field suggested by the bang-bang phenomenon).

The theory of Marchaud-Zaremba concerns only convex orientor fields, while the counter-domain $N(t, x)$ may be nonconvex. This leads to a difficulty which can be surmounted by introducing a suitable generalization of trajectory; that is the notion of quasitrajectory of an orientor field and of a control system.

It can be shown that the quasitrajectories of fields N, Q, E, of the system $S(f, C)$ and trajectories of field E form the same family. It follows that properties $\mathscr{P}_0 - \mathscr{P}_5$ of the convex field E hold true for system S provided trajectories are replaced by quasi-trajectories.

An implicit function theorem allows one to find the control u corresponding to a trajectory of a convex orientor field associated with $S(f, C)$.

The notion of the contingent derivative can be replaced in a convenient manner by the derivative in the classical sense. This leads to some generalizations. The method of estimation of the optimal time evident for orientor fields can be easily extended to system $S(f, C)$ applying Theorem 10. In the same way, some results of A. Bielecki [4] concerning the so called retract method for orientor fields can be generalized so as to apply to the control systems. By this method the problem of accessibility of a set by a trajectory and the asymptotic behaviour of trajectories can be treated.

One can observe the tendency of treating a control system as an orientor field in some other papers, e.g. of R. Kalman [7] and E. Roxin [12], though the authors do not refer directly to the theory of Marchaud-Zaremba. From a different point of view than ours the importance of the theory of Marchaud-Zaremba has been pointed out in a paper of E. A. Barbašin and Yu. I. Alimov [3].

The present article is mainly based on results of the author [14] to [24] and also on certain results of A. Pliś [10] and A. Turowicz [13].

1 Notation and definitions

We denote by:

$$R_n \text{ the real } n\text{-space},$$
$$x = (X_1, \dots, X_n) \text{ point of } R_n \text{ or } n\text{-vector},$$
$$O_n = (0, \dots, 0),$$
$$\emptyset \text{ the empty set}.$$

By orientor we mean a set of n-vectors, i.e. any subset of R_n if the points of R_n are meant to be vectors.

By

$$\operatorname{comp}(R_n), \quad (\operatorname{convex}(R_n))$$

we denote the collection of all nonempty compact (convex) subsets of R_n.

By

$$r(a, b) = |b - a| \quad \text{and} \quad r(a, B) = \inf_{x \in B} r(a, x)$$

we denote the distance of point a from point b and set B, respectively.

$V(A, k)$ denotes the closed neighbourhood of set A of radius k, i.e.

$$V(A, k) = \{x\colon r(x, A) \leqq k\}.$$
$$r^*(A, B) = \inf\{s\colon A \subset V(B, s),\ B \subset V(A, s)\}$$

is the well known Hausdorff's distance of two compact sets.

We put

$$|A| = r^*(O_n, A).$$

Let $T = (-\infty, +\infty)$, $W = T \times R_n$. A map $N(t, x)$ of W into $\operatorname{comp}(R_n)$ will be called an orientor field. If $N(t, x)$ reduces to a single point for each (t, x) then we have to do with a vector field, a special case of orientor field.

The continuity, upper and lower semicontinuity of orientor field $N(t, x)$ in the sense of Hausdorff is defined in a usual way.

By abs. cont (J) and mesbl (J) we denote the sets of all functions $x(t)$ absolutely continuous and measurable, respectively, on each compact contained in the interval J.

Let $x(t)$ be a function of t with values in R_n. Let g be an n-vector for which there exists a sequence $t_i \to t$, $t_i \neq t$ such that

$$\frac{x(t_i) - x(t)}{t_i - t} \to g \quad \text{as} \quad i \to \infty .$$

The set of all such g will be called the contingent derivative of $x(t)$ and we denote it by $D^* x(t)$.

A family Γ of functions $x(t)$ defined on T is said to be compact if each sequence $x_i(t) \in \Gamma$ contains a subsequence convergent to a function of Γ uniformly on every compact subset of T.

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Let Γ be a family of functions (or curves) $x = x(t)$. We assume that Γ possesses the following

Property $\mathscr{P}_0(\Gamma)$. Each $x(t)$ of Γ is defined and continuous in T and $x(t) \in R_n$ for $t \in T$. Each point $(t_0, x_0) \in W$ belongs to a curve of Γ.

Definition 1. *Let $A = (t_0, x_0) \in W$. By*

$$F(A, \Gamma)$$

we denote the family of $x(t) \in \Gamma$ such that $x(t_0) = x_0$, by

$$Z(A, \Gamma)$$

the union of the graphs of functions belonging to $F(A, \Gamma)$. The last set is called the zone of emission of point A with respect to Γ.

We put

$$L(A, \Gamma) = \text{boundary of } Z(A, \Gamma) .$$

We denote by $\Theta(k)$ and $\Lambda(k)$ the hyperplane $t = k$ and the halt space $t \geqq k$, respectively. We put

$$S(A, \Gamma, k) = Z(A, \Gamma) \cap \Theta(k) ,$$

$$Z_+(A, \Gamma) = \Lambda(t_0) \cap Z(A, \Gamma) ,$$

where $A = (t_0, x_0)$.

If $G \cap Z_+(A, \Gamma) \neq \emptyset$ then we will say that G is accessible from the point A by curves of Γ.

By the optimal time $t(A, G, \Gamma)$ of accessibility of the set G from the point A by curves of Γ we mean the minimum of k such that

$$B(k) = G \cap Z_+(A, \Gamma) \cap \Theta(k) \neq \emptyset .$$

Each point of $B(k)$ for $k = t(A, G, \Gamma)$ is called point of optimal accessibility of G from A by curves of Γ.

A point $A_3 = (t_3, x_3)$, $t_0 \leqq t_3$ is said to be peripherally accessible from $A = (t_0, x_0)$ if there exists a curve $x = x(t)$ of Γ such that $(t, x(t)) \in L(A, \Gamma)$ for $t_0 \leqq t \leqq t_3$ and $x(t_3) = x_3$.

Let $K \subset R_n$. Put

$$K^* = T \times K \quad (T = (-\infty, +\infty)) .$$

The set K is said to be accessible from point A if K^* is accessible from A.

Similarly we define the optimal time of accessibility, the points of optimal accessibility and the points of optimal peripheral accessibility of K from A.

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Definition 2. *We say that a family Γ of functions $x(t)$ has the property $\Delta(\Gamma)$ if property $\mathscr{P}_0$ as well as properties $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4, \mathscr{P}_5$ given below hold for Γ.*

Property $\mathscr{P}_1$. The family $F(A, \Gamma)$ is compact and the union

$$\bigcup_{A \in B} F(A, \Gamma)$$

is also compact for each compact $B \subset W$.

Property $\mathscr{P}_2$ (of Kneser). The set $S(A, \Gamma, k)$ is compact and connected for each point $A \in W$ and $k \in T$.

Property $\mathscr{P}_3$ (of Hukuhara). If $A = (t_0, x_0)$ and $A_1 = (t_1, x_1)$, $t_1 > t_0$ and $A_1 \in L(A, \Gamma)$ then A_1 is peripherally accessible from A.

Property $\mathscr{P}_4$ (optimal). If a closed set $Q \subset W$ is accessible from a point A along curves of Γ then it is accessible also in the optimal time.

Property $\mathscr{P}_5$ (optimal accessibility of point). If $x_1 \in R_n$ is accessible from a point $A = (t_0, x_0)$ then it is accessible in the optimal time and it is also accessible peripherally.

Remark. Let us point out that $\mathscr{P}_4$ follows from $\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$, and $\mathscr{P}_5$ from $\mathscr{P}_4$.

4 Orientor fields

Hypothesis $\mathscr{H}(N)$. For each $(t, x) \in W$, $N(t, x) \in \operatorname{comp}(R_n)$, $N(t, x)$ is bounded and continuous (see sec. 1) on W.

Hypothesis $\mathscr{H}_1(N)$. $N(t, x) \in \text{convex}(R_n)$ for each $(t, x) \in W$.

We now introduce two definitions of a trajectory of orientor field $N(t, x)$.

Definition 3. (of Marchaud [8]). *A function $x = x(t)$ defined in an interval J will be called a trajectory of $N(t, x)$ if*

$$D^*x(t) \subset N(t, x) \quad \textit{for each} \quad t \in J\,.$$

Definition 4. *A function $x = x(t)$ defined in an interval J will be called a trajectory of $N(t, x)$ if (see sec. 1)*

$$x(t) \in \text{abs. cont}(J)\,,$$
$$x'(t) \in N(t, x(t)) \quad (\text{a.e. } J)\,,$$

where $x'(t)$ denotes the derivative of $x(t)$ in the usual sense and the abbreviation a.e. *J means "almost everywhere in J".*

We have the following theorems (see [19]).

Theorem 1. *Assume $\mathscr{H}(N)$ and $\mathscr{H}_1(N)$. Then Definitions 3 and 4 are equivalent.*

Theorem 2. *Suppose $\mathscr{H}(N)$ and $\mathscr{H}_1(N)$ hold true and denote by $\Gamma(N)$ the family of trajectories of $N(t, x)$ which can be defined on the whole set T. Then (see sec. 2) we have property $\Delta(\Gamma(N))$; i.e. property $\mathscr{P}_i$ $(i = 0, 1, \ldots, 5)$ hold for $\Gamma(N)$.*

Proof. Owing to Marchaud [9] and Zaremba [27] properties $\mathscr{P}_0 - \mathscr{P}_3$ hold for $\Gamma(N)$. Thus $\mathscr{P}_4$ and $\mathscr{P}_5$ follow (see Remark of sec. 3).

5 Non-convex orientor fields. Quasitrajectories and strong quasitrajectories. Field $E(t, x)$ and field $Q(t, x)$ of type bang-bang

In what follows we always assume that $\mathscr{H}(N)$ holds, however we do not assume $\mathscr{H}_1(N)$.

By

$$E(t, x) = \text{env}\, N(t, x)$$

we denote the smallest convex set containing $N(t, x)$.

The smallest set $B(t, x)$ such that

$$B(t, x) \in \text{comp}(R_n) \quad \text{and} \quad \text{env}\, B(t, x) = \text{env}\, N(t, x)$$

will be called the tendor of $N(t, x)$ and we denote it by $Q(t, x)$, hence

$$Q(t, x) = \text{tend}\, N(t, x)\,.$$

We have the following theorems.

Theorem 3. *If $\mathscr{H}(N)$ holds then $\mathscr{H}(E)$ and $\mathscr{H}_1(E)$ also hold.*

Theorem 4. (See [20].) *If $\mathscr{H}(N)$ is satisfied then $Q(t, x)$ is lower semicontinuous on W.*

Definition 5. *A function $x = x(t)$ will be called a quasitrajectory of $N(t, x)$ if there exists a sequence $x_i(t)$ of functions such that*

$$\begin{gathered} x_i(t) \in \text{abs. cont}\,(T) \quad (i = 1, \ldots)\,, \\ x_i'(t) \textit{ are equibounded } (\text{a.e. } T)\,, \\ x_i(t) \to x(t) \quad \textit{for} \quad t \in T \quad \textit{as} \quad i \to \infty\,, \\ r(x_i'(t), N(t, x_i(t))) \to 0 \quad (\text{a.e. } T)\,. \end{gathered}$$

Function $x(t)$ will be called a strong quasitrajectory of $N(t, x)$ if there exists a sequence $x_i(t)$ of trajectories of $N(t, x)$ such that

$$x_i(t) \to x(t) \quad \textit{for} \quad t \in T\,.$$

Remark. Strong quasitrajectories are quasitrajectories, too. Also trajectories of an orientor field are quasitrajectories as well.

Definition 6. *Denote by $\{N\}$ the collection of all trajectories of $N(t, x)$, by $\{N\}^*$ the set of all quasitrajectories of $N(t, x)$.*

We have then (see [21])

Theorem 5. *Under the Hypothesis $\mathscr{H}(N)$ we have*

$$\{N\}^* = \{Q\}^* = \{E\}^* = \{E\}\,. \tag{1}$$

Remark. A proof, given in [21], that $\{E\} = \{Q\}^*$ is based strongly on Theorem 4 and on a result [10] of A. Pliś saying that a field $B(t)$ semicontinuous on T is continuous on T up to a set of arbitrarily small measure.

Theorem 6. *Suppose $\mathscr{H}(N)$ and let Γ be any of the four families of* (1). *Then Γ satisfies* $\Delta(\Gamma)$; *that is $\mathscr{P}_0 - \mathscr{P}_5$ hold for Γ.*

Proof. Theorem 6 is a consequence of Theorems 2, 3 and 5.

Theorem 7. *Under Hypothesis $\mathscr{H}(N)$ it can happen that $N(t, x)$ admits quasitrajectories which are not strong quasitrajectories.*

Theorem 7 follows from an example of A. Pliś [28].

Remark. The field $Q(t, x)$ is roughly speaking the smallest field $M(t, x) \subset N(t, x)$ which has under Hypothesis $\mathscr{H}(N)$ the same quasitrajectories as $N(t, x)$. It is closely connected to the bang-bang control method.

6 Estimation of the optimal time

Definition 7. *If*

(2) $$N_1(t, x) \subset N_2(t, x)$$

for $(t, x) \in W$ *then we will call field* $N_2(t, x)$ *a majorant of* $N_1(t, x)$ *and* $N_1(t, x)$ *a minorant of* $N_2(t, x)$.

We have an obvious

Theorem 8. *If, for* N_1 *and* N_2, $\mathscr{H}(N_1)$ *and* $\mathscr{H}(N_2)$ *holds, respectively and* (2) *is satisfied then the families of trajectories and quasitrajectories of fields* N_1 *and* N_2 *satisfy the following relations*

$$\{N_1\} \subset \{N_2\}, \quad \{N_1\}^* \subset \{N_2\}^* .$$

For emission zones with respect to families $\Gamma_1 = \{N_1\}^*$ and $\Gamma_2 = \{N_2\}^*$ we have

$$Z(A, \Gamma_1) \subset Z(A, \Gamma_2) .$$

Moreover, if a closed set $G \subset T \times R_n$ is accessible from A along quasitrajectories of N_1 then it is accessible also by quasitrajectories of N_2.

If we denote the corresponding optimal times of accessibility by t_1 and t_2, respectively, then $t_2 \leqq t_1$.

Further, if

(3) $$N_1(t, x) \subset N_2(t, x) \subset N_3(t, x) \quad \text{for} \quad (t, x) \in W$$

and N_3 satisfies $\mathscr{H}(N_3)$ and t_3 is the optimal time of accessibility corresponding to N_3, then $t_3 \leqq t_2 \leqq t_1$.

Remark. In order to estimate the optimal time of accessibility t_2 for a field N_2 one should look for two other fields N_1 and N_3 satisfying (3) and simple enough that it is easy to determine t_1 and t_3.

7 The case when an orientor field reduces to a vector field

Suppose that an orientor field $N(t, x)$ satisfies Hypothesis $\mathscr{H}(N)$ and, for each $(t, x) \in W$, $N(t, x)$ reduces to a single vector $g(t, x)$ which we write

$$N(t, x) = g(t, x) .$$

Then the families of trajectories $\{N\}$ and quasitrajectories $\{N\}^*$ are identical and each one of them is simply the family of all solutions of the system of n differential ordinary equations

$$x'(t) = g(t, x(t)) .$$

The properties of Kneser ($\mathscr{P}_2$) and of Hukuhara ($\mathscr{P}_3$) have been known before for such systems. Marchaud and Zaremba have extended those properties for convex orientor fields.

8 Control systems

Introduce the variables $x = (X_1, \ldots, X_n) \in R_n$ and $u = (U_1, \ldots, U_p) \in R_p$ and time t. Denote

$$T = (-\infty, +\infty), \quad W = T \times R_n, \quad Z = T \times R_n \times R_p = W \times R_p .$$

Let

$$f(t, x, u) = (F_1(t, x, u), \ldots, F_n(t, x, u))$$

be a mapping of Z into R_n. The variable u is called the control and x – position.

Hypothesis $\mathscr{H}(f, C)$. Function $f(t, x, u)$ is continuous and bounded on Z. The set

$$C(t, x) \in \operatorname{comp}(R_p)$$

is continuous and bounded on W in the sense of Hausdorff.

The control u and the set $C(t, x)$ are from the same space R_p.

Definition 8. *By a control system $S(f, C)$ we mean a pair: a function $f(t, x, u)$ and a field $C(t, x)$. $C(t, x)$ is called the control domain of $S(f, C)$.*

The set $N(t, x)$, for fixed (t, x), of all vectors $v = f(t, x, u)$ where u is taken from $C(t, x)$ will be called the control counterdomain of $S(f, C)$.

We have then an orientor field $N(t, x)$ defined on W, which we will call the orientor field associated to the control system $S(f, C)$.

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Definition 9. *A function $x = x(t)$ defined on an interval J is said to be a trajectory of $S(f, C)$, if $x(t) \in$ abs. cont (J) and if there exists a control function $u(t)$ such that*

$$x'(t) = f(t, x(t), u(t)) \quad (\text{a.e. } J),$$
$$u(t) \in \operatorname{mesbl}(J),$$
$$u(t) \in C(t, x(t)) \quad (\text{a.e. } J).$$

Definition 10. *A function $x = x(t)$ defined on J is said to be a quasitrajectory of $S(f, C)$ if there exist infinite sequences $x_i(t)$ and $u_i(t)$ $(i = 1, 2, \ldots)$ such that*

$$u_i(t) \in \operatorname{mesbl}(J),$$
$$x_i(t) \in \text{abs. cont}(J),$$
$$|x_i'(t)| \leqq M = \text{const} < +\infty \quad (\text{a.e. } J),$$

$$u_i(t) \in C(t, x_i(t)) \quad \text{(a.e. } J\text{)},$$
$$x_i'(t) - f(t, x_i(t), u_i(t)) \to 0 \quad \textit{as} \quad i \to \infty \quad \text{(a.e. } J\text{)},$$
$$x_i(t) \to x(t) \quad \textit{for} \quad t \in J .$$

Remark. Each trajectory of $S(f, C)$ is, of course, its quasitrajectory.

Theorem 9 (see [23]). *Under Hypothesis $\mathscr{K}(f, C)$, a sufficient and necessary condition for $x(t)$ to be a quasitrajectory of $S(f, C)$ on an interval $J = \langle a, a + h)$ is that there exists sequence $u_i(t) \in$ mesbl J such that*

$$u_i(t) \in C(t, x(t)) \quad \text{(a.e. } J\text{)},$$
$$x(t) - x(a) - \int_a^t f(s, x(s), u_i(s))\, \mathrm{d}s \to 0 \quad \textit{for} \quad t \in J .$$

Definition 11. *Every sequence $u_i(t)$ satisfying the conditions listed in Theorem 9 will be called an asymptotic control sequence corresponding to the quasitrajectory $x(t)$.*

Definition 12. *A function $x = x(t)$ will be called a strong quasitrajectory of $S(f, C)$ if there exists a sequence $x_i(t)$ $(i = 1, 2, \ldots)$ of trajectories of $S(f, C)$ such that $x_i(t) \to x(t)$.*

10 A connection between quasitrajectories of control system and those of orientor field

The families of trajectories and quasitrajectories of $S(f, C)$ are denoted by $\{f, C\}$ and $\{f, C\}^*$, respectively.

Remark. We have obviously $\{f, C\} \subset \{f, C\}^*$.

Theorem 10. (See [21], [22].) *Let $N(t, x)$ be the control counterdomain of $S(f, C)$ (see Definition 8) and let us introduce the following sets:*

$$E(t, x) = \operatorname{env} N(t, x), \quad Q(t, x) = \operatorname{tend} N(t, x) .$$

Under Hypothesis $\mathscr{K}(f, C)$, the orientor field $N(t, x)$ satisfies $\mathscr{H}(N)$ (see sec. 4) and we have

$$\{f, C\}^* = \{N\}^* = \{Q\}^* = \{E\}^* = \{E\} \tag{4}$$

(compare the notation of Theorem 5).

Theorem 11. *Assume Hypothesis $\mathscr{K}(f, C)$ and denote by Γ the family of quasi-trajectories of a control system $S(f, C)$, i.e.*

$$\Gamma = \{f, C\}^* .$$

Then the property $\Delta(\Gamma)$ holds; i.e. properties $\mathscr{P}_i$ ($i = 0, 1, \ldots, 5$) hold for Γ.

Properties $\mathscr{P}_3$, $\mathscr{P}_4$ concern some questions of the optimal control problem (in this case the control deals with quasitrajectories).

If we assume additionally that the control counterdomain $N(t, x)$ is convex then

$$\Gamma = \{f, C\}^* = \{f, C\},$$

i.e. each quasitrajectory is a trajectory, too. In that case the optimal properties $\mathscr{P}_3$, $\mathscr{P}_4$ concern trajectories.

Proof. Theorem 11 follows from Theorem 10 (see also [17]).

Theorem 12 (on bang-bang control method). *Assume Hypothesis $\mathscr{H}(f, C)$ and define the bang-bang kernel $C_1(t, x)$ of control domain $C(t, x)$ by the formula*

$$C_1(t, x) = \{u \colon u \in C(t, x), f(t, x, u) \in Q(t, x)\},$$

where $Q(t, x) = \operatorname{tend} N(t, x)$. *Then*

$$\{f, C\}^* = \{f, C_1\}^* .$$

Remark. By (4), the preceding results concerning estimates of the optimal time for orientor fields can be extended to control systems $S(f, C)$.

Remark. Owing to Property $\mathscr{P}_1$ one can get a result concerning the minimalization of the integral

$$\int_{t_0}^{t} m(t, x(t), u(t))\, \mathrm{d}t$$

by quasitrajectories of $S(f, C)$.

11 Relation between quasitrajectories and strong quasitrajectories

It follows from an example of A. Pliś that under Hypothesis $\mathscr{H}(f, C)$, (4) is not true if one replaces quasitrajectories by strong quasitrajectories.

A. Turowicz [13] has given some sufficient conditions that each quasitrajectory is strong quasitrajectory.

Let us notice, that the notion of strong quasitrajectory ("sliding regimes") was introduced independently and earlier by A. F. Filippov [6] under stronger hypotheses. However, this paper does not refer to the theory of Marchaud-Zaremba.

12 Method of elimination of the control variable — The inverse problem

Passing from a control system $S(f, C)$ (depending on the control u) to the associated orientor field $N(t, x)$ independent of u, we eliminate the control u.

The inverse problem consists of the following. Suppose we know a trajectory $x = x(t)$ of an orientor field $N(t, x)$ associated with a control system $S(f, C)$. We would like to find the corresponding control function $u(t)$. To do that we should find a measurable function $u(t)$ satisfying two conditions:

$$x'(t) = f(t, x(t), u),$$
$$u \in C(t, x(t)).$$

This problem requires a suitable implicit function theorem. We have dealt with a theorem of that type in [14]. A theorem of this kind is used also in [6] and [18].

The problem of the determination of an asymptotic control sequence $u_i(t)$ for a given quasitrajectory $x(t)$ of nonconvex field $N(t, x)$ can be dealt with in a similar way.

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The papers [15], [16], [1], [11] give a topological method for investigation of asymptotic efects in ordinary differential equations. The same papers also contain some sufficient conditions in order that a set B be accessible by at least one integral of the system issuing from another set A.

A. Bielecki [4] has extended this topological method for trajectories of convex orientor fields.

Owing to Theorem 10, Bielecki's results can be applied also to quasitrajectories of nonconvex orientor fields and control systems. Thus, by this method one may be able to solve the problem of accessibility (connected with properties $\mathscr{P}_4$, $\mathscr{P}_5$) and to study the asymptotic behaviour of quasitrajectories of control systems $S(f, C)$.

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Some of the above theorems also hold under more general assumptions [17] analogously as in the papers of Marchaud and Zaremba. One can also get some theorems of a local character.

15 Orientor fields satisfying Carathéodory's type assumptions

Hypothesis $\mathscr{H}^*(N)$. Orientor field $N(t, x) \in \text{comp}(R_n)$ is defined in $W = T \times R_n$, is Hausdorff continuous in x and measurable in t in the sense of Egorov. There exists $k(t) \geqq 0$ integrable in the interval T such that (see sec. 1)

$$|N(t, x)| \leqq k(t) \text{ a.e. } T, \quad x \in R_n.$$

Hypothesis $\mathscr{H}_1^*(N)$.

$$N(t, x) \in \text{convex}(R_n).$$

If we define a trajectory of $N(t, x)$ as in Definition 4, we have the following result mentioned in [25].

Theorem 13. *Under assumptions $\mathscr{H}^*(N)$ and $\mathscr{H}_1^*(N)$ the family Γ of trajectories of $N(t, x)$ has the property $\Delta(\Gamma)$ (see sec. 3).*

Remark. If only $\mathscr{H}^*(N)$ is assumed then we define quasitrajectory as in sec. 5. Relation (1) also holds in this case.

The idea of the proof consists in introducing an integral

$$\int_0^s B(t)\,\mathrm{d}t \tag{5}$$

where $B(t)$ is measurable in the sense of Egorov and (see sec. 1)

$$|B(t)| \leqq k(t)\,.$$

Such an integral is defined as a limit of algebraic sums of the form

$$c_1 A_1 \oplus c_2 A_2 \oplus \dots \oplus c_p A_p\,, \tag{6}$$

where c_i are non-negative constants and $A_i \in \text{convex}\,(R_n)$.

Above, by $A \oplus B$ we mean the set of all vectors $v = a + b$ where $a \in A$ and $b \in B$ and by cA we mean the set of all vectors $v = ca$, where $a \in A$.

The sum (6) and the integral (5) are convex sets.

We have (see notation of sec. 2)

$$S(A, \Gamma, k) = A \oplus \int_{t_0}^k B(t)\,\mathrm{d}t$$

where $\Gamma = \{B\}$ and A is a point.

Using the above formula one can get Kneser's property and in consequence Hukuhara's property similarly as in Marchaud-Zaremba theory.

We are indebted to Prof. Choquet for the information that an integral of the form (5) was considered by G. Mokobodzki (C. R. 1962) for another purpose.

An integral similar to (5) but defined in a different way appears also in a paper of E. A. Barbašin and Yu. I. Alimov [2].

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Suppose that orientor $E(t, x)$ is convex and put

$$B(t, x) = \text{boundary } E(t, x)\,.$$

Orientor field $B(t, x)$ determines a family of cones $M(t, x)$, which may be thought about as Monge's cones. Surface S which is tangent to $M(t, x)$ at each point $(t, x) \in S$ can be written in an implicit form $G(t, x) = \text{const}$. The function $G(t, x)$ satisfies a differential equation of the form

$$H(t, G_{X_1}, \dots, G_{X_n}) = 0$$

provided $M(t, x)$ is regular enough.

Applying Cauchy's characteristics method one can reduce the integration of this partial differential equation to a system of Hamilton's type, which of course does not contain the control variable. The characteristics issuing from a point A generate a cone-like surface R. Denote by Γ the family of trajectories of the orientor field $E(t, x)$. The emission zone $Z(A, \Gamma)$ has its boundary $L(A, \Gamma)$. Under suitable conditions of regularity, R and $L(A, \Gamma)$ are identical in a sufficiently small neighbourhood of A.

The Hukuhara type trajectories issuing from A can be obtained from Hamilton's system. The remarks enclosed in this section (see also [24]) can be found already in a slightly different form in papers of A. Marchaud. A similar point of view is presented also in a paper of R. Kalman [7].

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Added in proof:

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