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APPLICATION OF THE AVERAGING METHOD FOR THE SOLUTION OF BOUNDARY PROBLEMS FOR ORDINARY DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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The averaging method first appeared in space mechanics. The basic technique of the averaging method is to replace the right hand side parts of complex systems of differential equations by averaged functions, the latter not containing explicitly time and fast - changing parameters of the system.

The averaging method found a strict mathematical justification in the fundamental works of N.M. Krilov, N.N. Bogolubov and J.A.Mitropolsky [13] , [15] , [20] . This method reached its further development and generalization in [1] , [14] , etc.

The period after 1960 was one of vigorous development of the averaging method. At that time 7 monographs on the averaging method were published where a number of schemes were displayed for its application to the solution of initial problems. In this way naturally arose the question of the justification of the averaging method for the solution of boundary problems for ordinary differential equations. The first results concerning the justification of the averaging method for the solution of boundary problems for ordinary differential equations were obtained by D.D. Bainov in 1964, and from 1970 on the authors of this survey achieved a number of new results. Some of these results are exposed in the present paper.

1. Solution of boundary problems by means of the averaging method on the basis of asymptotics constructed for the Cauchy problem. In [14]V.M. Volosov proposes the general averaging scheme for the solution of the Cauchy problem for the system, as follows:

$$\begin{aligned} \dot{x} &= \varepsilon X(x,y,t,\varepsilon) = \varepsilon X_1(x,y,t) + \varepsilon^2 X_2(x,y,t) + \cdots \\ \dot{y} &= Y(x,y,t,\varepsilon) = Y_0(x,y,t) + \varepsilon Y_1(x,y,t) + \varepsilon^2 Y_2(x,y,t) + \cdots \end{aligned} \tag{1.1}$$

with initial condition $x(t_0)=x_0$, $y(t_0)=y_0$, where $x, X \in R_n$; $y, Y \in R_m$, while $\varepsilon > 0$ is a small parameter. In view of this the question of the possibility to apply the averaging method for solving boundary problems came to the fore. The paper [4] , namely, is devoted to the use of the averaging method to solve boundary problems for systems of the (1.1) type on the basis of asymptotics constructed for the Cauchy problem. An ordinary multipoint boundary problem and a multipoint boundary problem with boundary condition depending on several parameters are considered. Two theorems have been proved for each of these boundary problems. The first theorem points to conditions under which a formal asymptotics of the solution of the problem can be constructed. In the second theorem the existence and uniqueness of the solution of the boundary problem are proved. The paper [12] considers a boundary problem of the eigen-values for systems of ordinary differential equations with fast and slow variables. Theorems analogous to the ones in [4] have been proved.

2. Justification of the averaging method for the solution of two-point boundary problems for differential and integro-differential equations with fast and slow variables.

Consider the system of ordinary differential equations

(2.1) $\dot{x}(t) = \varepsilon X(t, x(t), y(t)), \quad \dot{y}(t) = Y(t, x(t), y(t))$

with boundary condition

(2.2) $x(0) = x_0 \; , \quad R[\lambda, y(0), y(T)] = 0$

where $x, X \in R_n; y, Y, R \in R_m; \lambda \in \Lambda \subset R_m; T = L\varepsilon^{-1}; L = \mathrm{const} > 0$, while $\varepsilon > 0$ is a small parameter. Together with the system (2.1), consider its degenerate system

(2.3) $\dot{y}(t) = Y(t, x, y(t)), \quad x = \mathrm{const}$

with boundary condition

(2.4) $R[\lambda, y(0), y(T)] = 0$.

Assume that the solution of the problem (2.3), (2.4) is known and has the form $y = \psi(t, x, \lambda, T)$. Then, if along the integral curves $y = \psi(t, x, \lambda, T)$ of the boundary problem (2.3), (2.4), where λ is considered as a vector parameter, there exists a non-dependent on λ mean value

(2.5) $\lim\limits_{T \to \infty} \frac{1}{T} \int\limits_0^T X(t, x, \psi(t, x, \lambda, T))\, dt = \overline{X}(x)$,

then the equation

(2.6) $\dot{\xi}(t) = \varepsilon \overline{X}(\xi(t))$

with initial condition

(2.7) $\xi(0) = X(0)$

will be called averaged equation of first approximation for the slow variables $x(t)$ of the system (2.1).

The following theorem for the proximity of the component $x(t)$ of the solution of boundary problem (2.1), (2.2) and the solution of the Cauchy problem (2.6), (2.7) holds.

THEOREM. Let us assume:

1. The functions $X(t,x,y)$ and $\frac{\partial}{\partial y}X(t,x,y)$ are continuous in the domain $\Omega(t,x,y)=\Omega(t)\times\Omega(x)\times\Omega(y)$, where $\Omega(t)=[0,\infty)$, $\Omega(x)$ and $\Omega(y)$ are certain open domains of the spaces R_n and R_m, resp.

2. In the domain $\Omega(t,x,y)$ the following inequalities are satisfied $\|X(t,x,y)-X(t,x',y')\|\leq\mu\|x-x'\|+\vartheta_1(t)\|y-y'\|$, $\|\frac{\partial}{\partial y}X(t,x,y)\|\leq\vartheta_2(t)$, where μ is a positive constant, while $\vartheta_i(t)$ $(i=1,2)$ is a continuous non-negative function.

3. The unique integral curve of boundary problem (2.3), (2.4) corresponding to some value of the parameter λ, passes through every point of the domain $\Omega(t,x,y)$, and besides,

a. This curve is definite and lies inside the domain $\Omega(y)$ for any $t\geq 0$.

b. The vector functions $\Psi(t,x,\lambda,T)$ and $\frac{\partial}{\partial T}\Psi(t,x,\lambda,T)$ are continuous along the set of variables t,x,λ,T and satisfy in the domain $\{\Omega(t,x)\times\Lambda=\Omega(t)\times\Omega(x)\times\Lambda,\ T\geq 0\}$ the inequalities

$$\|\Psi(t,x,\lambda,T)\|\leq K,\quad \|\frac{\partial}{\partial T}\Psi(t,x,\lambda,T)\|\leq g(t,T),$$

where K is a positive constant, while $g(t,T)$ is a continuous non-negative function.

4. The boundary problem (2.1), (2.2) has a unique continuous solution $\{x(t),y(t)\}$, whose component $y(t)$ is bounded ($\|y(t)\|\leq b=\mathrm{const}$). (In (2.2) λ means a certain fixed value of the parameter λ from the domain Λ.)

5. For $t\geq 0$ and $T\geq 0$ the functions $\vartheta_i(t)$ $(i=1,2)$ and $g(t,T)$ satisfy the conditions

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\vartheta_1(\tau)d\tau=0,\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t d\tau\int_0^\tau\vartheta_2(s)g(s,\tau)ds=0$$

6. For every $(x,\lambda)\in\Omega(x)\times\Lambda$ there exists a bound of (2.5) not depending on the parameter λ, and the boundary transition in (2.5) is accomplished uniformly with respect to the set $(x,\lambda)\in\Omega(x)\times\Lambda$. In the domain $\Omega(x)$ the function $\overline{X}(x)$ is continuous and satisfies the condition

$\|\overline{X}(x)\|\leq M$, $\|\overline{X}(x)-\overline{X}(x')\|\leq\nu\|x-x'\|$, where M and ν are positive constants.

7. The solution $\xi=\xi(t)$ of the Cauchy problem (2.6), (2.7) for any $t\geq 0$ is bounded ($\|\xi(t)\|\leq d=\mathrm{const}$) and lies in the domain $\Omega(x)$ together with some ρ-neighbourhood ($\rho=\mathrm{const}>0$).

Then, if $\{x(t),y(t)\}$ is a solution of the boundary problem (2.1), (2.2) and $\xi(t)$ is a solution of the Cauchy problem (2.6), (2,7) , then for any $\omega > 0$ and $L > 0$ such an $\varepsilon^0 > 0$ can be found that, for $0 \le \varepsilon \le \varepsilon^0$ on the cut $0 \le t \le L\varepsilon^{-1}$ the inequality $\| x(t) - \xi(t) \| < \omega$ will be satisfied.

PROOF. Introduce the function

(2.8) $$v(t,x) = \int_{\Omega(x)} \Delta_\alpha(x-x')\left\{ \int_0^t [X(\tau,x',\psi(\tau,x',\lambda,t)) - \overline{X}(x')]d\tau \right\} dx' ,$$

where the smoothing kernel $\Delta_\alpha(x)$ has the form $\Delta_\alpha(x) = A_\alpha(1-\frac{\|x\|^2}{\alpha^2})^2$ for $\|x\| \le \alpha$ and $\Delta_\alpha(x) = 0$ for $\|x\| > \alpha$, $\alpha = \mathrm{const} > 0$, while the positive constant A_α is determined by the condition $\int_{R_n} \Delta_\alpha(x)dx = 1$.

In view of the conditions of the theorem one can always construct such a monotonely decreasing function $\alpha(t)$ ($\alpha(t) \to 0$ for $t \to \infty$) that for every $x \in \Omega(x)$ the following inequality will hold

$$\| \frac{1}{t} \int_0^t [X(\tau,x,\psi(\tau,x,\lambda,t)) - \overline{X}(x)] d\tau \| \le \alpha(t) .$$

Then, for $t \ge 0$, for any points x , α whose neighbourhood belongs to the domain $\Omega(x)$, the following inequalities will hold

(2.9) $$\| v(t,x) \| \le t\alpha(t) , \quad \| \frac{\partial}{\partial x} v(t,x) \| \le I_\alpha t\alpha(t) ,$$

where $I_\alpha = \int_{R_n} \| \frac{\partial}{\partial x} \Delta_\alpha(x) \| dx$.

Estimate the expression

$$P(t,x) = \frac{\partial}{\partial t} v(t,x) - X(t,x,\psi(t,x,\lambda,t)) + \overline{X}(x) .$$

Since

$$\int_{\Omega(x)} \Delta_\alpha(x-x')dx' = \int_{\|x\| \le \alpha} \Delta_\alpha(x-x')\,dx' = 1 ,$$

then for $t \ge 0$, for any x , α whose neighbourhood belongs to the domain $\Omega(x)$, one obtains

(2.10) $$\| P(t,x) \| \le (\mu+\nu)\alpha + 2K\theta_1(t) + \int_0^t \theta_2(\tau) g(\tau,t)\,d\tau .$$

Set $\tilde{x}(t) = \xi(t) + \varepsilon v(t,\xi(t))$. According to the conditions of the theorem $\xi(t)$ lies in the domain $\Omega(x)$ together with the ρ - neighbourhood, and hence for $\alpha < \rho$ the estimate (2.9) holds for the function $v(t,\xi(t))$.

Set $A(\varepsilon) = \sup_{0 \le \tau \le L} \tau\alpha(\tau\varepsilon^{-1})$. Obviously, $A(\varepsilon) \to 0$ for $\varepsilon \to 0$ and the following inequality will hold on the segment $0 \le t \le L\varepsilon^{-1}$ if ε is sufficiently small:

(2.11) $$\| \varepsilon v(t,\xi(t)) \| \le \varepsilon t \alpha(t) \le A(\varepsilon) < \frac{1}{2} \min(\rho, \omega) .$$

Therefore, on the segment $0 \le t \le L\varepsilon^{-1}$, $\tilde{x}(t)$ belongs to the domain $\Omega(x)$ together with the ρ_1-neighbourhood ($0<\rho_1=const<\rho$) and $\|\tilde{x}(t)\| \le d_1$, $d_1 = const > 0$.

Consider the difference

$$(2.12)\qquad Q(t)=\frac{d\tilde{x}}{dt}-\varepsilon X(t,\tilde{x},y),$$

where $\tilde{x}=\tilde{x}(t)$, and $y=y(t)$ is a component of the solution $\{x(t), y(t)\}$ of the boundary problem (2.1), (2.2).

Taking into consideration (2.9), (2.10), one obtains

$$(2.13)\quad \|Q(t)\| \le \varepsilon(\mu+\nu)a+\varepsilon(3K+b)\theta_1(t)+\varepsilon^2(I_a M+\mu)t\alpha(t)+\varepsilon\int_0^t \theta_2(\tau)g(\tau,t)\,d\tau .$$

It is easily verified that on the segment $0 \le t \le L\varepsilon^{-1}$ the component $x(t)$ of the solution $\{x(t), y(t)\}$ of the boundary problem (2.1), (2.2) does not leave the domain $\Omega(x)$. Then on this segment one gets from (2.1) and (2.12)

$$\frac{d}{dt}\|x(t)-\tilde{x}(t)\| \le \varepsilon\mu\|x(t)-\tilde{x}(t)\|+\|Q(t)\|$$

whence, taking into account that $\tilde{x}(0)=x(0)$, one finds

$$(2.14)\quad \|x(t)-\tilde{x}(t)\| \le \int_0^t \|Q(\tau)\| \exp\{\varepsilon\mu(t-\tau)\}\,d\tau .$$

Introducing the functions

$$\delta_1(t)=\frac{1}{t}\int_0^t \theta_1(\tau)\,d\tau, \qquad \delta_2(t)=\frac{1}{t^2}\int_0^t \tau\alpha(\tau)\,d\tau$$

$$\delta_3(t)=\frac{1}{t}\int_0^t d\tau\int_0^\tau \theta_2(s)g(s,\tau)\,ds,\quad \delta_i(t)\to 0,\ t\to\infty\ (i=\overline{1,3}),$$

for the right hand side of the inequality (2.14) on the segment $0 \le t \le L\varepsilon^{-1}$ one finds the estimate

$$(2.15)\quad \int_0^t \|Q(\tau)\|\exp\{\varepsilon\mu(t-\tau)\}\,d\tau \le \exp\{\mu L\}\{(\mu+\nu)La+$$

$$+(3K+b)L\delta_1(L\varepsilon^{-1})+(I_a M+\mu)L^2\delta_2(L\varepsilon^{-1})+L\delta_3(L\varepsilon^{-1})\}.$$

From (2.11) and (2.15) it follows that if a and ε are sufficiently small ($0<a, \varepsilon \le \varepsilon^0$) then on the segment $0 \le t \le L\varepsilon^{-1}$ the inequality $\|x(t)-\xi(t)\| < \min(\rho,\omega)$ is satisfied. Thus, the theorem is proved.

In the papers [6] - [11], [16] - [19] several variants of the averaging method have been justified for the solution of two-point boundary problems for integro-differential equations with fast and slow variables.

Consider the system of integro-differential equations

$$\begin{aligned}\dot{x}(t) &= \varepsilon X\Big(t, x(t), y(t), \int_0^t \varphi(t,s,x(s),y(s))\,ds\Big)\\ \dot{y}(t) &= Y\Big(t, x(t), y(t), \int_0^t \varphi_1(t,s,x(s),y(s))\,ds\Big)\end{aligned} \tag{2.16}$$

with boundary conditions $x(0)=x_0$, $R[\lambda, y(0), y(T)]=0$, where $x, X \in R_n$; $y, Y, R \in R_m$; $\varphi \in R_p$; $\varphi_1 \in R_q$; $\lambda \in \Lambda \subset R_m$,

$T = L\varepsilon^{-1}$, $L = const > 0$, while $\varepsilon > 0$ is a small parameter.

Assume that the degenerate system with respect to (2.16)

$$\dot{y}(t) = Y\Big(t, x, y(t), \int_0^t \varphi_1(t,s,x,y(s))\,ds\Big) , \quad x = const \tag{2.17}$$

with boundary condition $R[\lambda, y(0), y(T)] = 0$, has a solution of the form

$$y = \psi(t, x, \lambda, T) , \quad x = const . \tag{2.18}$$

Several schemes of the averaging method are possible. Here is one of them.

Let along the integral curves (2.18), where λ is considered as a vector parameter, there exist mean values not depending on λ

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X(t, x, \psi(t,x,\lambda,T), u)\,dt = \overline{X}_1(x, u) ,$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(t, s, x, \psi(s,x,\lambda,T))\,ds = \overline{\varphi}_1(t, x) .$$

Then the equation

$$\dot{\xi}(t) = \varepsilon \overline{X}_1\Big(\xi(t), \int_0^t \overline{\varphi}_1(t, \xi(s))\,ds\Big)$$

with initial condition

$$\xi(0) = x(0)$$

will be called averaged equation of the first approximation for the slow variables $x(t)$ of the system (2.16).

This averaging scheme can be successfully applied when considering boundary problems for quasi-linear systems of the form

$$\begin{aligned}\dot{x}(t) &= \varepsilon\Big[\tilde{A}(x(t))y(t) + \tilde{B}(x(t)) + \int_0^t K(x(s))y(s)\,ds\Big]\\ \dot{y}(t) &= A(x(t))y(t) + B(x(t)) ,\\ \alpha y(0) &+ \beta y(T) = y_0 , \quad x(0) = x_0 ,\end{aligned} \tag{2.19}$$

where α is a diagonal matrix whose first p diagonal elements are units and the remaining $(m-p)$ are zeros; β is an analogous matrix whose first p elements are zeros and the rest are units;

$$A(x)=(a_{ij}(x))_1^m;\ \tilde{A}(x)=(\tilde{a}_{ij}(x))_{n,m};\ B(x)=(b_1(x),\dots,b_m(x));$$

$$\tilde{B}(x)=(\tilde{b}_1(x),\dots,\tilde{b}_m(x));\ K(x)=(K_{ij}(x))_{n,m};\ x\in R_n;\ y\in R_m,$$

while $\varepsilon>0$ is a small parameter.

Denote by $\nu_K=g_K(x)$ $(K=\overline{1,m})$ the eigen-values of the matrix $A(x)$, and by $\Omega(x)=(\omega_i^{(K)}(x))_1^m$ denote the matrix whose columns are composed by the components m of the linearly independent eigen-vectors of the matrix $A(x)$.

Under the assumption that in the considered domain $\Omega(x)$ the first p eigen-values of the matrix $A(x)$ have negative real parts and the remaining $(m-p)$ eigen-values have positive real parts, and under the assumption that the elementary divisors of the matrix $A(x)$ are simple and that $\operatorname{Det} M(x).\operatorname{Det} \mathcal{N}(x)\neq 0$,

where $M(x)=(\omega_i^{(j)}(x))_1^p$, $\mathcal{N}(x)=(\omega_\ell^{(s)}(x))_{p+1}^m$

in the paper [19] it is shown that the averaged equation of the system (2.19) has the form

$$\dot{\xi}(t)=\varepsilon\Big[\tilde{B}(\xi(t))-\tilde{A}(\xi(t))A^{-1}(\xi(t))B(\xi(t))-\\ -\int_0^t K(\xi(s))A^{-1}(\xi(s))B(\xi(s))\,ds\Big].$$

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