Jozef Brilla

Stability problems in mathematical theory of viscoelasticity

In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [46]--53.

Persistent URL: http://dml.cz/dmlcz/702202

STABILITY PROBLEMS IN MATHEMATICAL THEORY OF VISCOELASTICITY

J.Brilla, Bratislava

1. Introduction

The analysis of non-linear stability problems in the mathematical theory of viscoelasticity has only recently begun to attract attention and it will take some time before our understanding of these problems reaches the maturity similar to that of analysis of elastic stability problems.

We shall start from governing equations of large deflection theory of viscoelastic plates

$$(1.1)\quad \frac{h^3}{12} K_{ijkl}(D) w,_{ijkl} = K(D)(q + h\, \epsilon_{ik}\, \epsilon_{jl} w,_{ij} F,_{kl}) ,$$

$$\epsilon_{im}\, \epsilon_{jn}\, \epsilon_{kr}\, \epsilon_{ls} L_{mnrs}(D) F,_{ijkl} = - \frac{1}{2} \epsilon_{ik}\, \epsilon_{jl} L(D) w,_{ij} w,_{kl} ,$$

where

$$(1.2)\quad K_{ijkl}(D) = \sum_{\nu=0}^{r} K^{(\nu)}_{ijkl} D^{\nu} ,$$

$$(1.3)\quad L(D) = \sum_{\mu=0}^{s} L_{\mu} D^{\mu} ,$$

are polynomials in $D = \frac{\partial}{\partial t}$, $s = r$ or $r = s + 1$ and

$$(1.4)\quad K(D)\, [K_{ijkl}(D)]^{-1} = L(D)^{-1} L_{ijkl}(D) .$$

$K_{ijkl}(D)$, $K(D)$, $L_{ijkl}(D)$ and $L(D)$ are differential operators of linear viscoelasticity. w is the transverse displacement of the plate, F - the stress function, h - the thickness of the plate, q - transverse loading and ϵ_{ij} - the alternating tensor.

We use the usual indicial notation. Latin subscripts have the range of integers 1,2 and summation over repeated Latin subscripts is implied. Subscripts preceded by a coma indicate differentiation with respect to the corresponding Cartesian coordinates.

In the case of real materials it holds

$$(1.5)\quad K^{(\nu)}_{ijkl}(D)\, \varepsilon_{ij}\, \varepsilon_{kl} \geq 0$$

for arbitrary values of ε_{ij} and equality occurs iff $\varepsilon_{ij} = 0$ for all i,j . Further, the coefficients $K^{(\nu)}_{ijkl}$ are symmetric

$$(1.6)\quad K^{(\nu)}_{ijkl} = K^{(\nu)}_{jikl} = K^{(\nu)}_{ijlk} = K^{(\nu)}_{klij}$$

and polynomials (1.2)-(1.3) have real negative roots.

We assume that the domain of definition Ω is bounded with Lipschitzian boundary $\partial\Omega$. We shall consider the following boundary conditions

$$(1.7) \qquad w = \frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega ,$$

or

$$(1.8) \qquad w = 0 , \; K_{ijkl}(D) w,_{ij} \nu_{kn} \nu_{ln} = 0 \qquad \text{on } \partial\Omega ,$$

where $\nu_{kn} = \cos(x_k, n)$ and n is the outward normal to $\partial\Omega$ and

$$(1.9) \qquad \frac{\partial F}{\partial n} = \frac{\partial^3 F}{\partial n^3} = 0 \qquad \text{on } \partial\Omega$$

or

$$(1.10) \qquad \frac{\partial^2 F}{\partial n^2} = \frac{\partial^2 F}{\partial s \partial n} = 0 \qquad \text{on } \partial\Omega .$$

The initial condition may assume the form

$$(1.11) \qquad \frac{\partial^\nu w}{\partial t^\nu} = w_\nu \qquad (\nu = 0,1,2,\ldots,r-1)$$

and

$$(1.12) \qquad \frac{\partial^\nu F}{\partial t^\nu} = 0 \qquad (\nu = 0,1,2,\ldots,k-1)$$

where k is the order of the operator $L_{ijkl}(D)$.

Simultaneously we shall consider the integrodifferential equations

$$(1.13) \qquad \frac{h^3}{12} \int_0^t G_{ijkl}(t-\tau) \frac{\partial}{\partial\tau} w,_{ijkl}(\tau) d\tau = q + h \, \epsilon_{ik} \epsilon_{jl} w,_{ij} F,_{kl} ,$$

$$\int_0^t \epsilon_{im} \epsilon_{jn} \epsilon_{kr} \epsilon_{ls} J_{mnrs}(t-\tau) \frac{\partial}{\partial\tau} F,_{ijkl}(\tau) d\tau = -\frac{1}{2} \epsilon_{ik} \epsilon_{jl} w,_{ij} w,_{kl}$$

with boundary conditions (1.7)- (1.10) respectively, and the first initial conditions (1.11)-(1.12). These equations are the governing equations of the large deflection theory of viscoelastic plates of a material of Boltzmann type. $J_{ijkl}(t-\tau)$ is given by the inversion of the constitutive equations corresponding to $G_{ijkl}(t-\tau)$.

2. Linearized stability problems

When dealing with stability problems of time dependent processes we have to consider perturbations from equilibrium state. When considering perturbations that are extremely small (infinitesimal), we may feel justified in neglecting non-linear terms in (1.1) as compared to the linear ones.

We assume that the plate is subject to a system of two-dimensional stresses $\lambda h \sigma^o_{ij} = -\lambda N^o_{ij}$, where λ is a monotonically increasing factor of proportionality and the distribution of N^o_{ij} is prescribed. We put $q = 0$. In the resulting linearized stability theory we have

$$(2.1) \qquad \frac{h^3}{12} K_{ijkl}(D) w_{,ijkl} + \lambda N^o_{ij} w_{,ij} = 0$$

with boundary conditions (1.6) or (1.7) and initial conditions

$$(2.2) \qquad w = w_o \quad , \quad \frac{\partial^\nu w}{\partial t^\nu} = 0 \qquad (\nu = 1,2,\ldots,r-1) \quad .$$

Obviously the solution can be sought in the form

$$(2.3) \qquad w(x,y,t) = e^{\mu t} u(x,y) \quad .$$

Then the function $u(x,y)$ has to satisfy the partial differential equation

$$(2.4) \qquad \frac{h^3}{12} \sum_{\nu=0}^{r} K^{(\nu)}_{ijkl} \mu^\nu u_{,ijkl} + \lambda \sum_{\nu=0}^{s} K_\nu \mu^\nu N^o_{ij} u_{,ij} = 0 \quad .$$

Non-trivial solution for u exists only if the parameter μ assumes special values $\mu = \mu_n$ which are generalized eigenvalues of the linearized problem.

We shall assume that $r=s$, then we can write

$$(2.5) \qquad \sum_{\nu=0}^{r} \mu^\nu [\frac{h^3}{12} K^{(\nu)}_{ijkl} u_{,ijkl} + \lambda K_\nu N^o_{ij} u_{,ij}] = 0 \quad .$$

If Φ_n are eigenfunctions of our problem, it holds

$$(2.6) \quad \sum_{\nu=0}^{r} \mu_n^\nu [\frac{h^3}{12} K^{(\nu)}_{ijkl} (\Phi_{n,ij} , \Phi_{n,kl}) - \lambda K_\nu N^o_{ij} (\Phi_{n,i} , \Phi_{n,j})] = 0 \quad .$$

According to the assumptions the operator $\sum_{\nu=0}^{r} \mu^\nu K^{(\nu)}_{ijkl} (\cdot)_{,ijkl}$

is for positive values of μ positive definite. When dealing with stability problems we choose N^o_{ij} in such a way that

$\sum_{\nu=0}^{r} \mu^\nu K_\nu N^o_{ij} (\Phi_{,i} , \Phi_{,j})$ is positive. Then for sufficiently small

it holds

$$(2.7) \qquad \sum_{\nu=0}^{r} \mu^{\nu}\left[\frac{h^3}{12} K^{(\nu)}_{ijkl}(\Phi,_{ij}\ ,\Phi,_{kl}) - \lambda K_{\nu} N^{o}_{ij}(\Phi,_{i}\ ,\Phi,_{k})\right] \geq K^2||\Phi||^2 \ .$$

The polynomial on the left hand side has for sufficiently small λ positive coefficients and is a monotically increasing function for $\mu > 0$. Roots of this polynomial are then negative or have negative real parts. For an operator corresponding to real materials it can be proved that its roots are negative.

If the roots of (2.6) are simple and w_o in initial conditions (2.2) is considered as an initial perturbation, the solution of (2.1) can be written in the form

$$(2.8) \qquad w(x,y,t) = \sum_{n=1}^{\infty} \sum_{k=1}^{r} A_{nk} w_{on}\ \Phi_n(x,y) e^{-\mu_{nk} t} \ ,$$

where we have denoted the roots of (2.6) (which are real for real materials) by $-\mu_{nk}$,

$$(2.9) \qquad w_{on} = (w_o,\Phi_n)$$

and

$$(2.10) \qquad A_{nk} = \frac{\mu_{n1}\ \mu_{n2}\cdots\mu_{n(k-1)}\ \mu_{n(k+1)}\cdots\ \mu_{nr}}{(\mu_{nk} - \mu_{n1})\cdots(\mu_{nk} - \mu_{n(k-1)})(\mu_{nk} - \mu_{n(k+1)})\cdots(\mu_{nk} - \mu_{nr})}$$

If all $\mu_{nk} > 0$ the solution is stable. If at least one $\mu_{nk} < 0$ the solution is unstable.

The left hand side of (2.6) is a continuous function of λ and when λ increases it reaches critical values for which the roots successively change their signs.

In order to determine critical values of λ we have to analyse (2.6). We can write it in the form

$$(2.11) \qquad \sum_{\nu=0}^{r} \mu^{\nu} A_{n\nu}(\lambda) = 0 \ ,$$

which can be rewritten as

$$(2.12) \qquad \prod_{i=1}^{r} A_{nr}(\lambda)(\mu + \mu_{ni}(\lambda)) = 0 \ .$$

As it holds

$$(2.13) \qquad \prod_{i=1}^{r} \mu_{ni}(\lambda) = A_{nr}(\lambda)^{-1} A_{no}(\lambda)$$

and $A_{nr}(\lambda)$, $A_{no}(\lambda)$ are continuous functions of λ, the change of the signs of roots $\mu_{ni}(\lambda)$, assuming that they are not multiple, occurs at such values of λ, which satisfy equations

$$(2.14) \qquad A_{no}(\lambda) = \frac{h^3}{12} K^{(o)}_{ijkl}(\Phi_{n,ij}, \Phi_{n,kl}) - \lambda K_o N^o_{ij}(\Phi_{n,i}, \Phi_{n,j}) = 0$$

and

$$(2.15) \qquad A_{nr}(\lambda) = \frac{h^3}{12} K^{(r)}_{ijkl}(\Phi_{n,ij}, \Phi_{n,kl}) - \lambda K_r N^o_{ij}(\Phi_{n,i}, \Phi_{n,j}) = 0 .$$

Applying Laplace transform to (2.1) and making use of Tauber's theorem on limit values of Laplace transform we find out that these values are eigenvalues of the equations

$$(2.16) \qquad \frac{h^3}{12} K^{(o)}_{ijkl} w_{,ijkl}(\infty) + \lambda N^o_{ij} K_o w_{,ij}(\infty) = 0$$

and

$$(2.17) \qquad \frac{h^3}{12} K^{(r)}_{ijkl} w_{,ijkl}(0) + \lambda N^o_{ij} K_r w_{,ij}(0) = 0 .$$

When λ is an eigenvalue of (2.16) one of the roots (2.6) is equal to zero and when λ increases above this value one μ_{nk} becomes negative. When λ is an eigenvalue of (2.17) one of the roots (2.6) has to be equal to infinity.

We call eigenvalues of (2.16) critical values for infinite critical time and denote them by λ_{cr}. For $\lambda < \min \lambda_{cr}$ each $\mu_{nk} > 0$ and and the basic solution of (2.1) is stable. For $\lambda = \min \lambda_{cr}$ we have neutral stability and for $\lambda > \min \lambda_{cr}$ at least one $\mu_{nk} < 0$ and the basic solution is unstable with infinite critical time.

Eigenvalues of (2.17) are critical values of instant instability or critical values for finite critical time and we denote them by λ^o_{cr}. When λ reaches the value $\min \lambda^o_{cr}$ the plate becomes instantly unstable.

Now we have the following theorem:

Theorem 1. For $\lambda < \min \lambda_{cr}$, which is the minimum eigenvalue of (2.16), the basic solution of (2.1) is stable. For $\lambda = \min \lambda_{cr}$ this solution is neutral stable and for $\lambda > \min \lambda_{cr}$ the basic solution is unstable with infinite critical time. For $\lambda = \min \lambda^o_{cr}$, which is the minimum eigenvalue of (2.17) the basic solution of (2.1) becomes instantly unstable.

In the case of materials of Boltzmann type the corresponding critical values are eigenvalues of the equations

$$G_{ijkl}(\infty)w_{,ijkl}(\infty) + \lambda N^o_{ij}w_{,ij}(\infty) = 0 \tag{2.18}$$

and

$$G_{ijkl}(0)w_{,ijkl}(0) + \lambda N^o_{ij}w_{,ij}(0) = 0 \ . \tag{2.19}$$

3. Non-linear stability problems of viscoelastic plates

In the linear theory we assume that perturbations are arbitrarily small and in equations we neglect non-linear terms in the perturbation quantities as compared to the linear ones. In the case of instability with respect to infinitesimal perturbations we arrive at an apparent contradiction within the linearized theory, since we assume infinitesimal perturbations and find out that they grow without bounds. Therefore it is necessary to deal with non-linear analysis of stability problems. In studying non-linear problems we may find out that the magnitude of perturbation instead of growing without limit, tends to a finite value as time tends to infinity.

When dealing with nonlinear stability problems we restrict ourselves to an isotropic viscoelastic plate of a standard material. Thus we shall consider the generalized Karman equations

$$\begin{aligned} &K(1 + \alpha D)\Delta^2 w = h(1 + \beta D)(\lambda[F,w] + [f,w]) \ , \\ &(1 + \beta D)\Delta^2 f = -\tfrac{1}{2}E(1 + \alpha D)[w,w] \ , \end{aligned} \tag{3.1}$$

where

$$[f,w] = f_{,11}w_{,22} + f_{,22}w_{,11} - 2f_{,12}w_{,12} \ , \tag{3.2}$$

F is the particular solution of the linear equation

$$\Delta^2 F = 0 \tag{3.3}$$

with boundary conditions corresponding to given boundary loading, K is the bending stiffness of the plate, E is Young modulus and α, β coefficients of viscoelastic properties.

We consider the boundary conditions (1.7 - 10) for w and f and initial conditions

$$w(x,0) = w_0(x) \ , \quad f(x,0) = 0 \ . \tag{3.4}$$

Then we can prove the following theorem

Theorem 2. Critical values with infinite critical time and the corresponding stationary solutions are given by the equations

$$K\,\Delta^2 w = h(\lambda[F,w] + [f,w]) \ , \tag{3.5}$$

$$\Delta^2 f = -\frac{1}{2} E[w,w] \ .$$

Thus critical values with the infinite critical time of the non-linear problem are equal to those critical values of the linearized problem.

This result we can get directly considering the stationary solution or in the following way.

Equations (3.1) can be rewritten in the form:

$$K\left[\frac{\alpha}{\beta} \Delta^2 w + \left(1 - \frac{\alpha}{\beta}\right)\frac{1}{\beta} \int_0^t \Delta^2 w \, e^{-\frac{1}{\beta}(t-\tau)} d\tau - \frac{\alpha}{\beta} \Delta^2 w_0 \, e^{-\frac{1}{\beta}t}\right]$$

$$(3.6) \qquad = h\left(\lambda[F,w] + [f,w] - \lambda[F,w_0] \, e^{-\frac{1}{\beta}t}\right) ,$$

$$\frac{\beta}{\alpha} \Delta^2 f + \left(1-\frac{\beta}{\alpha}\right) \frac{1}{\alpha} \int_0^t \Delta^2 f \, e^{-\frac{1}{\alpha}(t-\tau)} d\tau = -\frac{1}{2}E\left([w,w] - [w_0,w_0] e^{-\frac{1}{\alpha}t}\right) .$$

Multiplyng $(3.6)_1$ scalarly by w and expressing $[w,w]$ from $(3.6)_2$ after some transformation we arrive at

$$K\{\frac{\alpha}{\beta}(\Delta w,\Delta w) + \left(1-\frac{\alpha}{\beta}\right)\frac{1}{\beta}(\Delta w, \int_0^t \Delta w \, e^{-\frac{1}{\beta}(t-\tau)} d\tau)\} - h\lambda(F,[w,w]) + \frac{2}{E}h\{\frac{\beta}{\alpha}(\Delta f,\Delta f)$$

$$+ \left(1 - \frac{\beta}{\alpha}\right)\frac{}{\alpha}(\Delta f, \int_0^t \Delta f \, e^{-\frac{1}{\alpha}(t-\tau)} d\tau)\}$$

$$(3.7) \qquad = K \frac{\alpha}{\beta}(\Delta w_0,\Delta w) e^{-\frac{1}{\beta}t} - h\lambda(F,[w_0,w]) \, e^{-\frac{1}{\beta}t} - h(f,[w_0,w_0]) e^{-\frac{1}{\alpha}t} .$$

As the total energy is finite, for $t = \infty$ we get

$$(3.8) \qquad K(\Delta w,\Delta w) - h\lambda(F,[w,w]) + \frac{2}{E} h(\Delta f,\Delta f) = 0 \ .$$

As $(\Delta f,\Delta f) \geq 0$, the nonzero solution of (3.8) exists iff λ is greater the critical values with the infinite critical time of the linearized problem, which are critical values of the non-linear problem (3.5) , too.

When there is no perturbation or $w_0 = 0$ then the right hand side of (3.7) is equal to zero and for $t = 0$ one arrive at

$$(3.9) \qquad K\alpha(\Delta w,\Delta w) - h\lambda\beta(F,[w,w] + \frac{2\beta^2}{E\alpha} h(\Delta f,\Delta f) = 0 \ .$$

The corresponding differential equations can be obtained from (3.6) . We get

$$
(3.10) \qquad \begin{aligned} K\alpha\Delta^2 w &= \beta h(\lambda[F,w]+[f,w]) \ , \\ \beta\Delta^2 f &= -\frac{1}{2} E\alpha[w,w] \ . \end{aligned}
$$

From (3.9) it is obvious that (3.9) and thus also (3.10) can have a non-zero solution iff

$$
(3.11) \qquad K\alpha(\Delta w,\Delta w) - h\beta\lambda(F,[w,w]) < 0
$$

and we can formulate the following theorem.

Theorem 3. The critical values for zero critical time, which are eigenvalues of (3.10) are critical values for zero critical time of the linearized problem.

The approximate solution of the problem which gives the exact values of critical values shows that critical values are bifurcation points. The bifurcation is continuous for $\lambda = \lambda_{cr}$ and discontinuous (by a jump) for $\lambda > \lambda_{cr}$.

References

[1] Brilla J.: Convolutional variational principles and stability of viscoelastic plates, Ing.Arch. 45 (1976), 273-280.

[2] Eckhaus W.: Studies in non-linear stability theory, Berlin--Heidelberg-New York, Springer 1965.

[3] Ržanicyn A.R.: Teorija polzučesti, Moskva 1968.

Author's address: Institute of Applied Mathematics and Computing Technique, Comenius University, Mlynská dolina, 816 31 Bratislava, Czechoslovakia