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BOUNDARY VALUE PROBLEMS
FOR SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

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1. A General Existence Theorem.

The lecture is devoted to the study of two-point boundary-value problems (abbreviation: BVP) for second order vector differential eq. of the form

(1.1) $$\ddot{x} = f(t,x).$$

Here $x = (x^1,\dots,x^n)^T$ is a n-dimensional column vector and the dot denotes differentiation with respect to the scalar variable t. We assume that f and its partial derivatives with respect to x are continuous functions of (t,x) on some open bounded set $\mathcal{P}$ in the (t,x)-space. The boundary conditions are assumed to be of the form

(1.2) $$x(0) = x_0, \quad x(1) = x_1.$$

To be more specific, we consider solutions $x(\cdot)$ of (1.1) on the interval $[0,1]$ which satisfy the condition

(1.3) $$(t,x(t)) \in \mathcal{P}, \quad 0 \le t \le 1,$$

and assume the prescribed values x_0 and x_1 respectively for $t=0$ and $t=1$ respectively.

In the first part of this lecture we present a general existence theorem which seems to be new in case of dimension $n > 1$. In case $n = 1$ the hypothesis of the theorem essentially amounts to the existence of so called upper and lower solutions. These are (scalar) functions $\alpha(\cdot)$, $\beta(\cdot)$ of class C^2 which satisfy the inequalities

(1.4) $$\ddot{\alpha}(t) > f(t,\alpha(t)), \quad \ddot{\beta}(t) < f(t,\beta(t)), \quad 0 < t < 1 .$$
$$\alpha(t) < \beta(t), \quad 0 \le t \le 1 .$$

It is well known that under these circumstances a solution $x(t)$ of the BVP (1.1), (1.2) exists, provided the boundary values x_0 and x_1 respectively are restricted to the intervals $[\alpha(0),\beta(0)]$, $[\alpha(1),\beta(1)]$ respectively. The existence of a solution is then established together with the a-priori estimate $\alpha(t)\le x(t)\le\beta(t)$ for $0\le t\le 1$ (see e.g. [3]).

In order to find a generalization of the above result to higher dimensions we observe that the two first of the relations (1.4) admit a simple geometric interpretation. Let us consider the region Ω in the (t,x)-plane given by

(1.6) $\Omega = \{t,x : 0 < t < 1 \ , \ \alpha(t) < x < \beta(t)\}$.

It is then easy to see that the said inequalities are equivalent with the following requirement:

(1.7) If a solution curve through a point $P_0 = (t_0,x_0) \in \partial\Omega$, where $0 < t_0 < 1$, is tangent to $\partial\Omega$ then it touches the set Ω from the exterior.

At this point some explanations seem to be in order. By a solution curve we mean a curve in the (t,x)-space (x need not be scalar from now on) which admits a parameteric representation $t \to (t,x(t))$, where $x(\cdot)$ is a solution of (1.1)."Tangent to $\partial\Omega$" means that the tangent to the curve at P_0 is in the tangent space to $\partial\Omega$ at P_0. "Touching from the exterior" means that $(t,x(t)) \notin \bar{\Omega}$ if $t \neq t_0$ and $|t-t_0|$ sufficiently small.

In passing we note that the statement (1.6) can also be phrased in this way: The set $\{t,x,\dot{x}) : (t,x) \in \Omega\}$ is an isolating block for the first order (2n-dimensional) system which is equivalent with (1.1).

We next write down two further statements which are evident in case $n=1$ (and if Ω is defined according to (1.6)) but which are substantial requirements in the general situation. For notational convenience we will use from now on the symbol Ω_t in order to denote the cross section $\{x : (t,x) \in \Omega\}$ of a given set Ω. Ω_t is a subset of the x-space; its interior relative to this space will be denoted by $\overset{o}{\Omega}_t$.

(1.8)
(i) Ω_t is convex, $\overset{o}{\Omega}_t$ is not empty.
(ii) There exists $q(\cdot) = (q^1(\cdot),\ldots,q^n(\cdot))^T$, each q^i being a function of class C^2 on $[0,1]$, such that $q(t) \in \overset{o}{\Omega}_t$ for every $t \in [0,1]$.

(1.9) $x_0 \in \Omega_0$, $x_1 \in \Omega_1$ (Ω_0, $\Omega_1 = \Omega_t$ for $t=0,1$).

The conditions (1.7)-(1.9) constitute the essential hypotheses of our general existence theorem. We add a further one which is of a more technical nature and can be relaxed somehow. It reduces the class of sets in the (t,x)-space which in the n-dimensional case will take the place of the special sets (1.6) to those which allow a simple analytic description.

(1.10) Ω is the intersection of finitely many regions of the form $\{t,x:\Phi(t,x)<0\}$. Each Φ is a scalar function of class C^2 on the whole (t,x)-space and satisfies
$k(t,x) \neq 0$, $H(t,x) > 0$ whenever $\Phi(t,x)=0$ and $(t,x)\in\partial\Omega$.

Here k and H respectively denote the n-dimensional vector and the symmetric $n \times n$-matrix respectively which are given by

(1.11) $k=(\Phi_{x^1},\dots,\Phi_{x^n})^T$, $H=(\Phi_{x^i x^j})$.

Note that the region Ω which is given (in case of n=1) by (1.6) falls in the category of sets which can be characterized in the form (1.10), (take $\Phi(t,x) = (x-\alpha(t))(x-\beta(t))$).

Theorem 1. Let Ω be an open subset of the (t,x)-space and let $\bar{\Omega} \subseteq \mathcal{P}$. Assume that (1.7)-(1.10) hold. Then there exists a solution $x(\cdot)$ of the BVP (1.1)-(1.3) with the property that $(t,x(t)) \in \bar{\Omega}$ for $0 \le t \le 1$.

A proof of Theorem1 - under slightly weaker hypotheses- can be found in the forthcoming paper [1] (cf. Theorem 5.2). It appears there in a setting which allows to treat by one and the same method various types of boundary conditions for differential eq. of the type (1.1) and also include certain cases where the right hande side of the differential eq. explicitely depends upon $\dot{x}$.

We conclude this section with a remark concerning the crucial hypothesis (1.7). Since it is of local nature one could expect that it can be replaced by conditions which do not involve a-priori knowledge of the solutions of (1.1). Indeed it is not difficult to convince oneself that the statement (1.7) is a consequence of the following requirement which has then to be met by every function Φ appearing in the analytic description (1.10) of Ω .

(1.12) $\Phi(t_o,x_o) = 0$ and $\dot{\Phi}(t_o,x_o,\dot{x}) = 0 \Longrightarrow \ddot{\Phi}(t_o,x_o,\dot{x}) > 0$.

Here $\dot{\Phi}$, $\ddot{\Phi}$ have to be understood as formal first and second order derivatives of Φ with respect to eq.(1.1) . $\dot{\Phi}$ is a linear, $\ddot{\Phi}$ a quadratic polynomial in $\dot{x}$.

Various alternative versions of (1.12) have been developed in [1]. The following one is convenient for our purposes. (1.12) can be inferred from an inequality of the form

(1.13) $k^T f - \rho\Phi_t - l^T H l + \Phi_{tt} > 0$,

where the scalar ρ and the vector l are subject to the linear constraint

(1.4) $2k_t + 2Hl = \rho k$

(for the definition of k and H see (1.11)). The argument in f,Φ,H,k is t_o,x_o.

2. An Application of Theorem 1.

Let there be given a positive definite symmetric $n\times n$ - matrix $Q(t)$ which is elementwise of class C^2 on $[0,1]$ and let

(2.1) $\varphi(t,x) = x^T Q(t) x$

be the corresponding quadratic form. Furthermore let $\tilde{x}(\cdot)$ be a solution of the differential eq. (1.1) which exists on $[0,1]$ and satisfies condition (1.3) (but which need not satisfy the boundary condition (1.2)). $\tilde{x}(\cdot)$ has to be regarded as fixed throughout this section. We adopt the following notation

(2.2) $\tilde{\varphi}(t,x) = \varphi(t,x-\tilde{x}(t)),\ \Omega_\delta = \{t,x:\ 0 < t < 1,\ \tilde{\varphi}(t,x) < \delta^2\}$.

Theorem 2. Let the matrix inequality

(2.3) $P(t,x) > 0,\quad (t,x) \in \mathcal{P},\quad 0 \le t \le 1,$

hold where

(2.4) $P(t,x) = Q(t)F(t,x) + F(t,x)^T Q(t) + \ddot{Q}(t) - 2\dot{Q}(t)Q(t)^{-1}\dot{Q}(t)$

and $F(t,x) = f_x(t,x)$ is the Jacobian matrix of f with respect to x. Furthermore let the positive number δ be chosen such that the inclusion

(2.5) $\bar{\Omega}_\delta \subseteq \mathcal{P}$

holds. Then the following statement is true. Whenever x_0, x_1 are such that

$$\tilde{\varphi}(0,x_0) \le \delta^2\ ,\quad \tilde{\varphi}(1,x_1) \le \delta^2$$

then the BVP (1.1) - (1.3) has a solution $x(\cdot)$ satisfying $\tilde{\varphi}(t,x(t)) \le \delta^2$ for all $t\in[0,1]$.

Proof. Most of the calculations which appear in the course of the proof are essentially the same as the ones used in the proof of Theorem 6.1 in [1]. Hence we skip here some details. On the other hand the procedure of the proof is simpler than in [1] and allows to dispose of the additional hypothesis $\tilde{\varphi}(0,x_0) = \tilde{\varphi}(1,x_1)$ which is required in [1] but which is superfluous. For the reader's convenience the proof is divided in two steps.

Step 1. We claim: For every compact subset $\mathcal{P}'$ of $\mathcal{P}$ there exists a positive number δ (depending upon $\mathcal{P}'$ only) such that the following statement holds true. Whenver $\Omega_\delta \subseteq \mathcal{P}'$ (regardless what $\tilde{x}(\cdot)$ is) then Ω_δ satisfies all hypotheses of Theorem 1.
It is clear that one has to verify the isolating block property of Ω_δ only. Since the boundary points (t,x) of Ω_δ with $0 < t < 1$ form the locus of the equation $\Phi(t,x) = \tilde{\varphi}(t,x)-\delta^2=0$ we may pursue the line described at the end of the previous section. Starting

with this particular Φ we determine H,k,ρ,l (cf. (1.11), (1.14)). It is easy to see by straightforward calculations that one arrives at the following result

$$(2.6)\qquad \begin{aligned} &k(t,x) = 2Q(x-\tilde{x}(t)),\ H(t,x) = 2Q, \\ &\rho = 0,\quad l=\dot{\tilde{x}}(t) - Q^{-1}\dot{Q}(x-\tilde{x}(t)) \ , \end{aligned}$$

where $Q = Q(t)$. For this choice of k and H the quantity on the left hand side of (1.13) turns out to be

$$(2.7)\quad k^T(f(t,x)-\ddot{\tilde{x}}(t)) + (x-\tilde{x}(t))^T[\ddot{Q} - 2\dot{Q}Q^{-1}\dot{Q}](x-\tilde{x}(t)).$$

Let us now consider the function

$$p(t,x,\tilde{x}) = 2(x-\tilde{x})^TQ(t)[f(t,x)-f(t,\tilde{x})]$$

which is defined and of class C^1 on a neighborhood of the set

$$(2.8)\quad \{t,x,\tilde{x} : 0 \le t \le 1,\ (t,x) \in \mathcal{P}',\ (t,\tilde{x}) \in \mathcal{P}'\}.$$

The function p and all its partial derivatives with respect to $x,\tilde{x}$ vanish whenever $x=\tilde{x}$. One easily recognizes that the second order term in the Taylor-expansion at $x=\tilde{x}$ can be expressed in terms of the Jacobian $F = f_x$ as

$$(2.9)\quad (x-\tilde{x})^T[Q(t)F(t,\tilde{x}) + F(t,\tilde{x})^TQ(t)](x-\tilde{x}).$$

It is then clear, by standard arguments, that the difference between p and the above quadratic form is of order $\mathcal{O}(\|x-\tilde{x}\|^2)$. On the other hand it follows from (2.6) and (2.7) that the left hand side of (1.13) can be identified with

$$(2.10)\quad p(t,x,\tilde{x}) + (x-\tilde{x})^T[\ddot{Q} - 2\dot{Q}Q^{-1}\dot{Q}](x-\tilde{x}) \quad \text{for} \quad \tilde{x} = \tilde{x}(t) .$$

Replacing p in this formula by the quadratic form (2.9) turns (2.10) into the quadratic form $(x-\tilde{x})^TP(t,\tilde{x})(x-\tilde{x})$ (for the definition of P see (2.4). It follows now from what was said in connection with (2.9) that the latter differs from the expression (2.10) by an error term of order $\mathcal{O}(\|x-\tilde{x}\|^2)$. Since, according to the hypothesis of our theorem, the matrix P is positive on the compact set $\mathcal{P}'$ one can find $\delta > 0$ such that

$$p(t,x,\tilde{x}) + (x-\tilde{x})^T[\ddot{Q} - 2\dot{Q}Q^{-1}\dot{Q}](x-\tilde{x}) > 0$$

whenever $t,x,\tilde{x}$ belong to the set (2.8) and $\|x-\tilde{x}\| \le \delta$. For this choice of δ the sets Ω_δ will then have all properties listed in Theorem 1 and hence the statement of Theorem 2 becomes an immediate consequence of what we found in Section 1.

<u>Step 2.</u> The general case – δ is now subject to the condition $\bar{\Omega}_\delta \subseteq \mathcal{P}$ only – can be handled in precisely the same way as in [1] (cf. the last subdivision of the proof of Theorem 6.1). We take $\bar{\Omega}_\delta$ as $\mathcal{P}'$ and choose a sequence of intermediate points $x_0^{(i)}$, $x_1^{(i)}$, $i=1,\dots,M-1$, respectively between $\tilde{x}(0),x_0$ and $\tilde{x}(1),x_1$ respectively such that the distance between two consecutive points is not

bigger than the δ which we elaborated in the first part of the proof. This allows us to solve for each $i=1,2,\dots,M$ the BVP

$$(2.11)_i \quad \ddot{x} = f(t,x), \quad x(0) = x_0^{(i)}, \quad x(1) = x_1^{(i)}$$

if we let the solution $x^{(i-1)}(\cdot)$ of the preceding problem $(2.11)_{i-1}$ play the role of $\tilde{x}(\cdot)$ and if we take the given $\tilde{x}(\cdot)$ as $x^{(o)}(\cdot)$. The solution of $(2.11)_M$ will then have all desired properties. Thereby the proof of the theorem is complete.

Corollary. The conclusion of Theorem 2 remains valid if one has instead of (2.3), (2.4) a matrix inequality of this form

$$(2.12) \quad Q(t)F(t,x) + F(t,x)^T Q(t) + \ddot{Q}(t) - 2R(t) > 0$$

where R satisfies the following condition. One can find a positive symmetric matrix $\hat{Q}(t)$ of type $m' \times m'$, for some $m' \geq m$, such that

$$(2.13) \quad \hat{Q}(t) = \begin{pmatrix} Q(t) & * \\ * & * \end{pmatrix}, \quad \dot{\hat{Q}}(t)\hat{Q}(t)^{-1}\dot{\hat{Q}}(t) = \begin{pmatrix} R(t) & * \\ * & * \end{pmatrix}$$

where the asterisks denote submatrices of the types $m \times m'$, $m' \times m$ and $m \times m$ respectively. As before the inequality has to hold for all $(t,x) \in \mathcal{P}$ with $0 \leq t \leq 1$; one also has to assume that $\hat{Q}$ is elementwise of class C^2 on $[0,1]$.

Proof. We may assume without loss of generality that $m' > m$. Let y be a new state variable which is of dimension $m'-m$ and let us consider the m'-dimensional system

$$(2.14) \quad \ddot{x} = f(t,x) - \mu Q^{-1}(t) Q_1(t) y, \quad \ddot{y} = \mu y .$$

where $\mu = \mu(t,x)$ is a scalar function and Q_1 is the submatrix in the right upper corner of $\hat{Q}$. We now treat (2.14) as a single system of the form $\ddot{\hat{x}} = \hat{f}(t,\hat{x})$ where $\hat{x}$ is the pair (x,y). It is then not difficult to convince oneself that one can choose μ in such a way that the inequality

$$(2.15) \quad \hat{Q}\hat{F} + \hat{F}^T\hat{Q} + \ddot{\hat{Q}} - 2\dot{\hat{Q}}\hat{Q}^{-1}\dot{\hat{Q}} > 0$$

holds on the set $\{t,x,y : (t,x) \in \mathcal{P}, y=0\}$. This is a consequence of the hypothesis (2.12). Hence it is clear, in view of (2.5), that the inequality (2.15) will also hold on the closure of the set

$$(2.16) \quad \hat{\mathcal{P}} = \{t,x,y : (t,x) \in \Omega_\delta, \ \|y\| < \varepsilon\} ,$$

provided $\varepsilon > 0$ is sufficiently small. We wish to apply Theorem 2 to the diff. eq. (2.14) with $\hat{\mathcal{P}}$ and $\hat{\tilde{x}}(t) = (\tilde{x}(t),0)$ playing the roles of $\mathcal{P}$ and $\tilde{x}(t)$ respectively. It follows now by inspection that a solution $\hat{x}(\cdot)$ of (2.14) which assumes the boundary values

$$(2.17) \quad \hat{x}(0) = (x_0,0), \quad \hat{x}(1) = (x_1,0)$$

is necessarily of the form $\hat{x}(t) = (x(t),0)$, where $x(\cdot)$ is a solution of the BVP (1.1), (1.2). Therefore we need for the present situation not the full analogy of condition (2.5) but we can get along with the weaker requirement that the intersection of $\overline{\hat{\Omega}}_\delta$ with the set $\{t,\hat{x}=(x,y) : y = 0\}$ belongs to $\hat{\mathcal{P}}$. This however is true, in view of (2.16) and we can infer the existence of a solution $\hat{x}(t) = (x(t),0)$ of the eq. (2.14) which satisfies the boundary conditions (2.17) as well as the inequality

$$(\hat{x} - \tilde{\hat{x}})^T\hat{Q}(\hat{x} - \tilde{\hat{x}}) = (x - \tilde{x})^T Q(x - \tilde{x}) \leq \delta^2$$

for all $t \in [0,1]$. Thereby the corollary is proved.

3. Uniqueness and Continuous Dependence.

In this section we will present a statement concerning uniqueness and continuous dependence of the solutions. It will bring out the importance of the conditions (2.12) and (2.13) for the study of the two-point boundary value problem. Related results have been established previously by Hartman ([2], Chapter XII, cf. in particular Theorem 4.3).

Theorem 3. Let the cross-sections $\mathcal{P}_t = \{x : (t,x) \in \mathcal{P}\}$ be convex, for $0 \leq t \leq 1$, and assume that matrix relations of the form (2.12), (2.13) hold for all $(t,x) \in \mathcal{P}$. Then for any two solutions $x(\cdot),\tilde{x}(\cdot)$ of eq. (1.1) which are such that the corresponding curves remain in $\mathcal{P}$ for $0 \leq t \leq 1$ the following statement holds true: The function

$$\rho(t) = (x(t) - \tilde{x}(t))^T Q(t)(x(t)-\tilde{x}(t)) = \tilde{\varphi}(t,x(t))$$

satisfies $\rho(t) \leq \text{Max}\,(\rho(0),\rho(1))$ for $0 \leq t \leq 1$.

Proof. We first consider the linear case, i.e. we assume that $f(t,x)$ has the form $F(t)x$ and that we have

$$(3.1) \quad Q(t)F(t) + F(t)^T Q(t) + \ddot{Q}(t) - 2R(t) > 0$$

for all $t \in [0,1]$. It follows then from Theorem 2 and its corollary that there exists, for a r b i t r a r y choice of x_0,x_1, a solution $x(\cdot)$ of the differential equation $\ddot{x}=F(t)x$ which satisfies the boundary conditions (1.2) and which has the properties stated in the conclusion of the theorem with respect to an arbitrary solution $\tilde{x}(t)$ (take as $\mathcal{P}$ a sufficiently large region of the (t,x)-space and choose $\delta^2 = \text{Max}(\rho(0),\rho(1))$). Solving the BVP (1.1), (1.2) in the linear case however amounts to solving n linear equations in n unknowns. Indeed one can determine the solution $x(\cdot)$ by setting up a system of linear equations for $\dot{x}(0)$. This system has the simple form $Ax_0 + B\dot{x}(0) = x_1$ and is clearly solvable for arbitrary x_1 if and only if $\det B \neq 0$. Hence for the linear BVP condition (3.1) guarantees uniqueness. This in turn implies that the conclusion of the theorem holds for an arbitrary pair of solutions of the linear differential eq. $\ddot{x} = F(t)x$, since $x(\cdot)$ then can be identified with a solution whose existence together with the a-priori-estimate follows from Theorem 2.

For the proof in the non-linear case we use the same argument as Hartman (loc.cit.). Let $z(t) \equiv x(t)-\tilde{x}(t)$, then $z(t)$ is solution of the linear differential eq. $\ddot{z} = \tilde{F}(t)z$, where

$$\tilde{F}(t) = \int_0^1 F(t,x(s,t))ds, \quad x(s,t) = sx(t) + (1-s)\tilde{x}(t).$$

Because of the convexity of $\mathcal{P}_t$ the relation (2.12) holds for $x = x(s,t)$ and $0 \le s \le 1$, $0 \le t \le 1$. If we integrate with respect to s we obtain the relation (3.1) with $\tilde{F}$ instead of F. Hence the conclusion of the theorem follows from our previous considerations.

We add a further remark concerning the dependence of the solutions of the BVP (1.1) - 1.3) from the boundary data x_0, x_1. Under the hypothesis of Theorem 3 they are Lipschitz-continuous functions of x_0, x_1 as can be seen immediately from the statement of the theorem. It turns out however that they are even continuously differentiable functions of x_0, x_1. This can easily be established from the following observation. As a consequence of (2.12) the variational eq. $\ddot{y} = F(t,x(t))y$ along a given solution $x(\cdot)$ falls into the category of linear differential eqs. satisfying condition (3.1). From what we found out in the course of the last proof the desired result follows then by a standard application of the implicit function theorem.

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