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ASYMPTOTIC METHODS FOR SINGULARLY PERTURBED LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Introduction

Take a Banach space $\mathfrak{X}$ with the norm $\|\cdot\|$ and a singularly perturbed differential equation

$$(1) \qquad \frac{dz(t)}{dt} = T(t)z(t) + m(t)$$

where $T(t)$ is a time-dependent linear operator, $m(t)$ a given function, and ϵ a small positive parameter. Such equations were analyzed by Krein [1] who proved that the zero order asymptotic solution consisting of three parts: inner, outer, and intermediate, converges uniformly to the exact solution. In [2] the author found the uniformly convergent asymptotic solution of any fixed order containing only outer and inner parts matched together by a proper choice of initial conditions. Such a procedure was already used by Vasil'eva and Butuzov [3] for systems of ordinary differential equations.

From the practical point of view, much more interesting are systems of singularly perturbed differential equations

$$(2) \qquad \begin{aligned} \epsilon \frac{dx(t)}{dt} &= A(t)x(t) + P(t)y(t) + q(t) ; \\ \frac{dy(t)}{dt} &= Q(t)x(t) + B(t)y(t) + r(t) , \end{aligned}$$

which were analyzed by the author in the zero order approximation in [4] and in any fixed order approximation in [5] . The results can be also applied to a single differential equation in a Banach space

$$(3) \qquad \frac{dz(t)}{dt} = \frac{1}{\epsilon} A\, z(t) + Bz(t) + m(t)$$

such that the singularly perturbed operator A has an eigenvalue at the origin contrary to the conditions which have to be satisfied for (1) . If A and B are bounded operators then the Banach space can be split into a direct sum of two subspaces and (3) written as a system of differential equations which can be treated similarly as (2) . In particular, one can prove the uniform convergence of the asymptotic solution to the exact one for the initial value problem for the equation (3) . The detailed analysis will be published elsewhere.

The asymptotic expansion method for differential equations with singularly perturbed operators having an eigenvalue at the origin was first applied by Hilbert to the Boltzmann equation in kinetic theory. The presented results give a rigorous justification for the Hilbert approach in case of bounded operators.

As an example, the linear Boltzmann equation for neutrons in a discretized form is considered and the asymptotic expansion method used to derive the diffusion equation.

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Hilbert asymptotic expansion method

Let A be a bounded linear operator from $\mathfrak{X}$ into itself with zero as its semisimple isolated eigenvalue. The corresponding finite-dimensional eigenspace $\mathcal{N}$ consists only of such elements $y \in \mathfrak{X}$ that $Ay = 0$. The space $\mathfrak{X}$ can be represented as a direct sum

$$\mathfrak{X} = \mathcal{N} + \mathcal{M}$$

where both $\mathcal{N}$ and $\mathcal{M}$ are invariant subspaces of A and A is one-to-one from $\mathcal{M}$ into itself [8, Section 148] . If P is the projector from $\mathfrak{X}$ onto $\mathcal{N}$ and $Q = I - P$ then

$$P\mathfrak{X} = \mathcal{N} \quad ; \quad Q\mathfrak{X} = \mathcal{M}.$$

The spectrum of A, except the point at the origin, is assumed to be located in the left half-plane and bounded away from the imaginary axis so that

$$\alpha = \inf \operatorname{Re} \lambda < 0 \,, \qquad \lambda \in \operatorname{Sp}A;\ \lambda \neq 0 \,. \tag{4}$$

Thus the uniformly continuous semigroup $G(t)$ generated by QAQ taken as an operator from $\mathcal{M}$ into itself satisfies the inequality

$$\|G(t)\| \leqslant M \cdot \exp(\alpha t) \quad ; \quad 0 \leqslant t < \infty$$

where M is a constant.

If B is a bounded operator in $\mathfrak{X}$ and $m(t)$ a function with values from $\mathfrak{X}$ n times continuously differentiable on $[0, t_0]$, then with the assumed properties of A, B the equation (3) has on $[0, t_0]$ a unique, strongly differentiable solution $z(t)$ for the initial condition

$$z(0) = \theta$$

where θ is an arbitrary element of $\mathfrak{X}$.

Define new functions

$$Qz(t) = v(t) \;; \qquad Pz(t) = w(t) \;;$$
$$Qm(t) = q(t) \;; \qquad Pm(t) = r(t) \;;$$
$$Q\theta = \mu \;; \qquad P\theta = \eta$$

and transform (3) into an equivalent system of equations

$$(5)\quad \begin{aligned} \frac{dv(t)}{dt} &= QAQv(t) + QBQv(t) + QBPw(t) + q(t) ; \\ \frac{dw(t)}{dt} &= PBQv(t) + PBPw(t) + r(t) \end{aligned}$$

with the corresponding initial condition

$$(6)\qquad v(0) = \mu \; ; \quad w(0) = \eta .$$

The functions $v(t)$ and $q(t)$ have values from $\mathcal{M}$ and $w(t)$ and $r(t)$ from $\mathcal{N}$. The operators in (5) are defined accordingly so that, for instance, QBP is an operator from $\mathcal{N}$ into $\mathcal{M}$. In deriving (5) it was taken into account that for any y

$$PAy = APy \equiv 0.$$

The system of equations (5) is analogous to (2) so that an approach similar to that presented in [5] can be applied to obtain the asymptotic solution of any fixed order to (5) with the initial condition (6).

Let the outer asymptotic solution of order n be defined as

$$\bar{v}^{(n)}(t) = \sum_{k=0}^{n} \epsilon^k \bar{v}_k(t) \quad ; \quad \bar{w}^{(n)}(t) = \sum_{k=0}^{n} \epsilon^k \bar{w}_k(t) \; ;$$

then $\bar{v}_k(t)$ and $\bar{w}_k(t)$ satisfy the equations

$$(7)\quad \begin{aligned} &QAQ\bar{v}_k(t) + QBQ\bar{v}_{k-1}(t) + QBP\bar{w}_{k-1}(t) + \delta_{1k}q(t) = \frac{d\bar{v}_{k-1}(t)}{dt} \; ; \\ &\frac{d\bar{w}_k(t)}{dt} = PBQ\bar{v}_k(t) + PBP\bar{w}_k(t) + \delta_{ok}\, r(t) \quad ; \; k = 0,1,\dots n, \\ &\qquad\qquad \bar{v}_{-1}(t) = \bar{w}_{-1}(t) \equiv 0. \end{aligned}$$

Similarly, if the inner asymptotic solution of order n is defined as

$$\tilde{v}^{(n)}(\tau) = \sum_{k=0}^{n} \epsilon^k \tilde{v}_k(\tau) \quad ; \quad \tilde{w}^{(n)}(\tau) = \sum_{k=0}^{n} \epsilon^k \tilde{w}_k(\tau) \quad ; \quad \tau = t/\epsilon \; ;$$

then $\tilde{v}_k(\tau)$ and $\tilde{w}_k(\tau)$ satisfy the equations

$$(8)\quad \begin{aligned} &\frac{d\tilde{v}_k(\tau)}{d\tau} = QAQ\tilde{v}_k(\tau) + QBQ\tilde{v}_{k-1}(\tau) + QBP\tilde{w}_{k-1}(\tau) \; ; \\ &\frac{d\tilde{w}_k(\tau)}{d} = PBQ\tilde{v}_{k-1}(\tau) + PBP\tilde{w}_{k-1}(\tau) \quad ; \; k = 0,1,\dots n; \\ &\qquad\qquad \tilde{v}_{-1}(\tau) = \tilde{w}_{-1}(\tau) \equiv 0. \end{aligned}$$

The algorithm of solving (7) and (8) consists of the following steps:

(i) $\tilde{w}_k(\tau)$ is found by solving the second equation in (8) and using the condition that $\lim_{\tau\to\infty} \tilde{w}_k(\tau) = 0$; for $k = 0$ it gives simply $\tilde{w}_0(\tau) \equiv 0$;

(ii) $\bar{v}_k(t)$ is eliminated from the equations (7) and the resulting equation for $\bar{w}_k(t)$ is solved with the initial condition

$$\bar{w}_k(0) = \delta_{ok} - \tilde{w}_k(0) ;$$

(iii) $\bar{v}_k(t)$ is calculated from $\bar{w}_k(t)$ with the first equation in (7) ;

(iv) the first equation in (8) is solved with

$$\tilde{v}_k(0) = \delta_{ok} - \bar{v}_k(0) .$$

The functions $\tilde{v}_k(\tau)$ and $\tilde{w}_k(\tau)$ decay in the norm exponentially with time as $\exp(\alpha\tau)$ where α is defined in (4) .

If the asymptotic solution of order n is taken as

$$v^{(n)}(t) = \bar{v}^{(n)}(t) + \tilde{v}^{(n)}\left(\frac{t}{\epsilon}\right) \quad ; \quad w^{(n)}(t) = \bar{w}^{(n)}(t) + \tilde{w}^{(n)}\left(\frac{t}{\epsilon}\right) ,$$

then $\{v^{(n)}(t) , w^{(n)}(t)\}$ tends uniformly on $[0, t_0]$ to the exact solution $\{v(t) , w(t)\}$ of (5) and (6) faster than ϵ^n.

Application of the Hilbert method in neutron transport theory

A singularly perturbed differential equation containing the operator with an eigenvalue at the origin was first considered by Hilbert in the kinetic theory (see e.g. [6]). The Hilbert expansion of the Boltzmann equation, later modified by Chapman and Enskog, has played an important role in statistical physics. However, it is based essentially on intuitive grounds. Only in the case of the linearized Boltzmann equation for special initial conditions it was rigorously analyzed by Grad [7] .

The results obtained by the author and presented in previous section indicate the convergence of the Hilbert expansion for differential equations in Banach spaces but an obvious disadvantage of the analysis is that the operators have to be bounded. Nevertheless, the obtained results can be applied to unbounded operators provided a mollifying procedure is used. An example of such an approach will be given in this section.

The behavior of neutrons in reactor systems is described by the linear Boltzmann equation. In practical situations, this equation is by far too complicated to be treated directly so that various approximate models have to be applied. One of the most widely used is the diffusion approximation. In the literature there are several ways of deriving the neutron diffusion equation from the Boltzmann equation but so far the Hilbert expansion method was not utilized.

Take a simplest possible reactor system consisting of an infinite slab of thickness a and assume that all neutrons have the same speed which will be taken as equal to unity. The Boltzmann equation for such a system is

$$(9)\quad \frac{\partial z}{\partial t} = -\xi \frac{\partial z}{\partial \rho} - (\beta_a(\rho) + \beta_s(\rho))\, z(\rho, \xi, t) + \beta_s(\rho)\int_{-1}^{1} d\xi'\, h_s(\rho; \xi, \xi')\, z(\rho, \xi', t) + m(\rho, \xi, t) \;;\; 0 \leqslant \rho \leqslant a; \; -1 \leqslant \xi \leqslant 1 \;;\; 0 \leqslant t < \infty .$$

Here $z(\rho, \xi, t)$ is the neutron distribution function dependent on the position variable ρ , angular variable ξ , and time t; $\beta_a(\rho)$ and $\beta_s(\rho)$ are absorption and scattering frequencies, respectively; $h_s(\rho; \xi, \xi')$ is the scattering kernel assumed to have the form

$$(10)\quad h_s(\rho; \xi, \xi') = \sum_{l=0}^{\infty} \frac{2l+1}{2}\, b_l(\rho)\, p_l(\xi)\, p_l(\xi'),$$

where $p_l(\xi)$ are Legendre polynomials and

$$(11)\quad b_0(\rho) \equiv 1 \;;\; b_l(\rho) < 1 \;;\; 0 \leqslant \rho \leqslant a \;;\; l = 1,2,\ldots .$$

The solution to (9) is assumed to be periodic so that the boundary condition to be satisfied by $z(\rho, \xi, t)$ is

$$(12)\quad z(0, \xi, t) = z(a, \xi, t) \;;\; -1 \leqslant \xi \leqslant 1;\; 0 \leqslant t < \infty .$$

The equation (9) is supplemented by the initial condition

$$(13)\quad z(\rho, \xi, 0) = \theta(\rho, \xi);\; 0 \leqslant \rho \leqslant a \;;\; -1 \leqslant \xi \leqslant 1.$$

It can be shown [8] that the equation (9) with the conditions (12) and (13) has a unique solution in the Hilbert space of square integrable functions.

The unbounded differential operator in (9) can be replaced by its finite difference counterpart if the interval $[0,a]$ is covered by the mesh of points $0 = \rho_0 < \rho_1 < \ldots < \rho_J = a$ and all the functions of ρ are replaced by vectors or matrices whose components are values of the relevant functions taken at $\rho_1 \ldots \rho_J$. The boundary condition (12) is accounted for by identifying the values of $z(\rho, \xi, t)$ at ρ_0 and ρ_J.

The derivative in (16) is replaced by the matrix

$$D = \begin{vmatrix} 0 & \varkappa_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -\varkappa_1 \\ -\varkappa_2 & 0 & \varkappa_2 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -\varkappa_3 & 0 & \varkappa_3 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & -\varkappa_{J-1} & 0 & \varkappa_{J-1} \\ \varkappa_J & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -\varkappa_J & 0 \end{vmatrix}$$

where $\varkappa_1 = (\rho_2 - \rho_J)^{-1}$, $\varkappa_J = (\rho_1 - \rho_{J-1})^{-1}$, and $\varkappa_j = (\varrho_{j+1} - \varrho_{j-1})^{-1}$; j = 2, ... J-1. All the remaining operators in (9) are in their discretized form diagonal.

In the matrix notation one can write (9) as

$$(14)\qquad \frac{dz}{dt} = -\xi Dz(\xi,t) - (\beta_a + \beta_s) z(\xi,t) + \beta_s \int_{-1}^{1} d\xi\, h_s(\xi,\xi') z(\xi',t) + m(\xi,t) ;$$

and the initial condition (13) as $z(\xi,0) = \Theta(\xi)$.

As the space $\mathfrak{X}$ we shall take the product $\mathfrak{X} = \mathcal{L} \times \dots\dots \times \mathcal{L}$ of Hilbert spaces $\mathcal{L}$ of functions square integrable over $[-1,1]$.

Define the operator A as

$$(Ax)(\xi) = \beta_s \Big(\int_{-1}^{1} d\xi'\, h(\xi,\xi')\, x(\xi') - x(\xi) \Big) ; \quad x \in \mathfrak{X}$$

with

$$h(\xi,\xi') = h_s(\xi,\xi') + \frac{3}{2}(I - b_1)\; p_1(\xi)\, p_1(\xi')$$

and B as

$$(Bx)(\xi) = -\xi Dx(\xi) - \beta_a x(\xi) - \frac{3}{2}\beta_s (I - b_1)\, p_1(\xi) \int_{-1}^{1} d\xi'\, p_1(\xi') x(\xi') ; \quad x \in \mathfrak{X} ;$$

where I is the unit matrix.

With the above definitions, introducing the coefficient $\frac{1}{\varepsilon}$ in front of A one can write (14) in the form (3) and apply the Hilbert expansion method. The null space $\mathcal{N}$ is now the sum $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1$, where $\mathcal{N}_0 = \mathcal{N}_0^{(0)} \times \dots\dots \times \mathcal{N}_0^{(0)}$; $\mathcal{N}_1 = \mathcal{N}_1^{(0)} \times \dots\dots \times \mathcal{N}_1^{(0)}$; and $\mathcal{N}_0^{(0)}$ and $\mathcal{N}_1^{(0)}$ are linear manifolds in $\mathcal{L}$ spanned by the elements $\frac{1}{2} p_0(\xi)$ and $\frac{3}{2} p_1(\xi)$, respectively. The projectors corresponding to $\mathcal{N}_0$ are given by the formulas

$$(P_0 x)(\xi) = \frac{1}{2} \int_{-1}^{1} d\xi'\, p_0(\xi') x(\xi') ; \quad (P_1 x)(\xi) = \frac{3}{2} p_1(\xi) \int_{-1}^{1} d\xi'\, p_1(\xi') x(\xi') .$$

From (11) it follows that the spectrum of A excluding the point at origin satisfies the requirement (4).

Defining new functions

$$P_0 z(t) = \varphi(t) \; ; \quad P_1 z(t) = \xi\, \chi(t) \; ;$$
$$P_0 m(t) = m_0(t) \; ; \quad P_1 m(t) = \xi\, m_1(t) \; ;$$
$$P_0 \theta = \theta_0 \; ; \quad P_1 \theta = \xi\, \theta_1$$

and applying the procedure of the previous section one gets in the zero order approximation the following outer asymptotic equations

$$(15)\qquad \frac{d\,\overline{\varphi}_0(t)}{dt} = -\beta_a \overline{\varphi}_0(t) - \frac{1}{3} D\, \chi_0(t) + m_0(t) \; ;$$

$$\frac{d\,\overline{\chi}_o(t)}{dt} = -D\,\overline{\varphi}_o(t) - \beta_t\,\overline{\chi}_o(t) + m_1(t)$$

and the corresponding initial conditions $\overline{\varphi}_o(0) = \Theta_o$; $\overline{\chi}_o(0) = \Theta_1$. The matrix β_t is defined as $\beta_t = \beta_a + \beta_s(I - b_1)$.

The equations (15) are a discretized representation of the equations obtained from the first order spherical harmonics approximation as applied to (9) with $\overline{\varphi}_o(t)$ as the neutron density and $\overline{\chi}_o(t)$ as the neutron current. Obviously, they could be improved with higher order outer solutions together with inner solutions according to the general scheme developed in the previous section.

The equations (15) have a different structure from the diffusion equation. In fact, if β_a and β_t are constant matrices, then by eliminating $\overline{\chi}_o(t)$, (23) can be reduced to the second order differential equation being a discretized version of the telegraph equation.

To obtain the discretized diffusion equation from (15) one has to introduce another small parameter, say, δ in front of the derivative $\frac{d\,\overline{\chi}_o(t)}{dt}$ and apply the asymptotic expansion method as for the equation (2) . As the result in the zero order approximation one gets the outer asymptotic equations with respect to both small parameters ϵ and δ which after eliminating the current $\overline{\overline{\chi}}_{oo}(t)$ lead to the discretized diffusion equation

$$\frac{d\,\overline{\overline{\varphi}}_{oo}(t)}{dt} = \frac{1}{3}\,D\,\beta_t^{-1}\,D\,\overline{\overline{\varphi}}_{oo}(t) - \beta_a\,\overline{\overline{\varphi}}_{oo}(t) + \overline{\overline{m}}(t)$$

with the initial condition

$$\overline{\overline{\varphi}}_{oo}(0) = \Theta_o$$

and

$$\overline{\overline{m}}(t) = m_o(t) - \frac{1}{3}\,\beta_t^{-1}\,Dm_1(t)\ .$$

Thus it is seen that the repeated application of the asymptotic expansion method to the Boltzmann equation with respect to two different small parameters ϵ and δ leads in the zero order outer approximation to the diffusion equation, at least in the discretized representation. However, the physical justification for introducing both small parameters into (14) and (15) is not so convincing as in the case of the Boltzmann equation for gases. The more detailed discussion of this problem will be performed elsewhere.

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