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ENERGETIC ESTIMATES ANALOGOUS TO THE SAINT-VENANT PRINCIPLE
AND THEIR APPLICATIONS
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In 1855 Saint-Venant [1] formulated a principle which is of an exceptional importance in the theory of elasticity as well as in its applications in the construction mechanics. During the last hundred years numerous studies have been devoted to the Saint-Venant principle and to the clarification of conditions of its applicability. A strict mathematical formulation of the Saint-Venant principle together with its justification for cylindrical bodies was given by Toupin [2] in 1965 and for arbitrary twodimensional bodies by Knowles [3]. A survey of investigations concerning this problem is found in Gurtin's paper [4].

The Saint-Venant principle can be expressed in the form of an a priori energetic estimate of the solution of the system of equations of the elasticity theory. It was found that estimates of this type can be established for wide classes of partial differential equations and systems. Theorems of the Phragmen-Lindelöf type, existence and uniqueness theorems for solutions of boundary value problems in both bounded and unbounded domains in the class of functions with unbounded energy integrals, theorem on the behavior of solutions in the neighborhood of non-regular points of the boundary (in the neighborhood of angles, ribs etc.) and in the neighborhood of infinity can be obtained as consequences of the energetic estimates which express the Saint-Venant principle. A number of such results was obtained in [5] - [17].

As the simplest example let us consider the Saint-Venant principle for the Laplace equation in a domain $\Omega$ of a special shape.

**Theorem 1.** Let a bounded domain $\Omega$ from the class $C^1$ coincide for $|x_n| < T$ with a cylinder $\{x: x' \in \Omega', -T < x_n < T\}$ where $x = (x_1, \ldots, x_n), x' = (x_1, \ldots, x_{n-1}), T = \text{const.}, \Omega'$ is a domain in the space $\mathbb{R}_{x}^{n-1}$. Assume that $u \in C^2(\Omega) \cap C^1(\Omega')$,

(1) $\Delta u = f$ in $\Omega$, $\frac{\partial u}{\partial y} |_{\partial \Omega} = \psi$, $\Delta u \equiv \sum_{j=1}^{n} u_{x_j}x_j$,

where $f \equiv 0$ in $\Omega_T$, $\psi \equiv 0$ on $\partial \Omega \cap \partial \Omega_T$ and, moreover,

(2) $\int_{\Omega^+} f dx - \int_{\partial \Omega \cap \partial \Omega^+} \psi ds = 0$, $\int_{\Omega^-} f dx - \int_{\partial \Omega \cap \partial \Omega^-} \psi ds = 0$,
where $\Omega_{\tau} = \Omega \cap \{ x : |x_n| < \tau \}$, $\tau = \text{const} > 0$, $\tau \leq T$, $\Omega^+ = \Omega \cap \{ x : x_n > T \}$, $\Omega^- = \{ x : x_n < -T \}$, $\partial \Omega$ is the boundary of $\Omega$, $\gamma$ is the direction of the unit outer normal to $\partial \Omega$. Then

$$
\int_{\Omega_{\tau_0}} |\nabla u|^2 \, dx \leq \exp \left\{ -2 \lambda^2 (\tau_1 - \tau_0) \right\} \int_{\Omega_{\tau_1}} |\nabla u|^2 \, dx,
$$

where $|\nabla u|^2 = \sum_{j=1}^n u_j^2$, $\tau_0$, $\tau_1 = \text{const} > 0$, $\tau_0 < \tau_1 \leq T$.

(4) $\lambda = \inf_{v \in M} \left\{ \int_{\Omega'} \sum_{j=1}^{n-1} v_j^2 \, dx' \left[ \int_{\Omega'} v^2 \, dx' \right]^{-1} \right\}$,

$M$ is the family of all functions $v(x')$ continuously differentiable on $\Omega'$ which satisfy the condition

$$
\int_{\Omega'} v(x') \, dx' = 0.
$$

Proof. With regard to (1) we obtain for $-T < a < T$

$$
\int_{\Omega^+} \Delta u \, dx = \int_{\Omega_n \{ x : x_n < a \}} \psi \, dx + \int_{\partial \Omega \cap \partial \Omega^+} u \chi_n \, dx',
$$

This together with the conditions (2) implies

$$
\int_{x_n = a} u \chi_n \, dx' = 0, \quad -T < a < T.
$$

Let $S_\tau^+ = \{ x : x_n = \tau \}$, $S_\tau^- = \{ x : x_n = -\tau \}$, $S_\tau = S_\tau^+ \cup S_\tau^-$. According to the Green formula, we have for arbitrary positive $\tau \leq T$

$$
0 = \int_{\Omega_{\tau}} u \Delta u \, dx = - \int_{\Omega_{\tau}} |\nabla u|^2 \, dx + \int_{S_\tau^+} uu \chi_n \, dx' - \int_{S_\tau^-} uu \chi_n \, dx'.
$$

Taking into account the relation (6), we conclude that for any constants $C_\tau^+$ and $C_\tau^-$

$$
\int_{\Omega_{\tau}} |\nabla u|^2 \, dx = \int_{S_\tau^+} (u + C_\tau^+) u \chi_n \, dx' - \int_{S_\tau^-} (u + C_\tau^-) u \chi_n \, dx'.
$$
Consequently

\[
\int_{\Omega_\tau} |\nabla u|^2 dx \leq \left[ \int_{S^+_{\tau}} (u + C^+)^2 dx' \right]^{1/2} \left[ \int_{S^-_{\tau}} u_n^2 dx \right]^{1/2} + \\
+ \left[ \int_{S^-_{\tau}} (u + C^-)^2 dx \right]^{1/2} \left[ \int_{S^-_{\tau}} (u_n^2)^2 dx' \right]^{1/2} .
\]

Let us choose the constants \( C^+ \) and \( C^- \) so that

\[
\int_{S^+_{\tau}} (u + C^+)^2 dx' = 0 , \quad \int_{S^-_{\tau}} (u + C^-)^2 dx' = 0 .
\]

Then the inequality (7) and the relation (8) imply that

\[
\int_{\Omega_\tau} |\nabla u|^2 dx \leq \lambda^{-1/2} \left[ \int_{S^+_{\tau}} \sum_{j=1}^{n-1} u_{x_j}^2 dx \right]^{1/2} \left[ \int_{S^-_{\tau}} u_{x_n}^2 dx \right]^{1/2} + \\
+ \lambda^{-1/2} \left[ \int_{S^-_{\tau}} \sum_{j=1}^{n-1} u_{x_j}^2 dx' \right]^{1/2} \left[ \int_{S^-_{\tau}} u_{x_n}^2 dx' \right]^{1/2} \leq \\
\leq \frac{1}{2} \lambda^{-1/2} \int_{S_{\tau}} \left( \sum_{j=1}^{n-1} u_{x_j}^2 + u_{x_n}^2 \right) dx' = \frac{1}{2} \lambda^{-1/2} \int_{S_{\tau}} |\nabla u|^2 dx' .
\]

Set \( F(\tau) = \int_{\Omega_\tau} |\nabla u|^2 dx , \quad 0 \leq \tau \leq T . \) We obtain from (8) that

\[
F(\tau) \leq \frac{1}{2} \lambda^{-1/2} \frac{dF}{d\tau} .
\]

Multiplying this inequality by \( \exp \{ -2 \lambda^{1/2} \tau \} \cdot 2 \lambda^{1/2} \) and integrating from \( \tau_0 \) to \( \tau_1 \) we obtain the inequality (3). The theorem is proved.

The conditions (2) for a membrane correspond in the Saint-Venant principle for an elastic body to the condition that the forces acting at the ends are statically equivalent to zero. The number \( \lambda \) defined by the conditions (4), (5) equals to the first non-zero eigenvalue of the Neumann boundary value problem.
It is easily seen that \( \lambda = \pi^2/1^2 \) for \( n = 2 \), where 1 is the length of the interval \( S_x \). The following theorem of the Phragmen-Lindelöf type (a uniqueness theorem) for the solution of the Neumann problem in an infinite cylinder \( \Omega \) is a consequence of the estimate (3).

**Theorem 2.** Let \( \Omega = \{ x: x' \in \Omega', -\infty < x_n < +\infty \} \), \( \Delta u = 0 \) in \( \Omega \), \( \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega} = 0 \), \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) and let for a constant \( b \)

\[
\int_{x_n=b} u(x') \, dx' = 0.
\]

Then \( u \equiv \text{const} \) in \( \Omega \) provided there is a sequence \( R_j \rightarrow \infty \) with

\[
2 \lambda \int_{\Omega_{R_j}} |\nabla u|\,dx \leq C(R_j) \exp \left\{ 2 \lambda \frac{1}{2} R_j \right\},
\]

where \( C(R_j) \rightarrow 0 \) for \( R_j \rightarrow \infty \).

This theorem is an immediate consequence of Theorem 1.

The constant \( 2 \lambda \frac{1}{2} \) which appears in the exponential function in the inequalities (3) and (10) is the best possible, i.e. it cannot be replaced by a greater constant. This is demonstrated by the following example. Let \( v(x') \) be a non-trivial solution of the problem (9) corresponding to the first non-zero eigenvalue \( \lambda \). Then

\[
\int_{\Omega'} v(x')\,dx' = 0.
\]

Put \( u(x) = v(x') \exp \left\{ \frac{1}{2} \lambda x_n \right\} \). The function \( u(x) \) satisfies all assumptions of Theorem 2 except the condition (10). The inequality (10) holds for \( u(x) \) provided \( C(R_j) = \text{const} > 0 \) and hence \( u \equiv \text{const} \). Indeed, for any \( R > 0 \) we have

\[
\int_{\Omega_R} |\nabla u|\,dx = C_1 \int_{-R}^{R} \exp \left\{ 2 \lambda \frac{1}{2} x_n \right\} \, dx_n \leq C_2 \exp \left\{ 2 \lambda \frac{1}{2} R \right\}, \quad C_1, C_2 = \text{const}.
\]
Analogously to the proof of Theorem 1 we can prove an estimate of the type (3) and a Phragmen-Lindelöf theorem for solutions of the Dirichlet problem for the Laplace equation. The following assertion holds.

**Theorem 3.** Let \( \Omega \) be the domain defined in Theorem 1, \( \Delta u = f \) in \( \Omega \), \( u|_{\partial \Omega} = \psi \) with \( f \equiv 0 \) in \( \Omega_T \), \( \psi \equiv 0 \) on \( \partial \Omega \cap \partial \Omega_T \), \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \). Then \( u(x) \) satisfies the inequality (3) with

\[
\lambda = \inf_{v \in M_1} \left\{ \frac{\int_{\Omega'} \sum_{j=1}^{n-1} v_{x_j}^2 \, dx'}{\left[ \int_{\Omega'} v^2 \, dx' \right]^{-1}} \right\},
\]

\( M_1 \) being the family of all functions \( v(x') \) continuously differentiable in \( \Omega' \) and such that \( v|_{\partial \Omega'} = 0 \).

Let us notice that no conditions are put on \( f \) in \( \Omega \setminus \Omega_T \) and on \( \psi \) in \( \partial \Omega \setminus \partial \Omega_T \) in Theorem 3 in contradistinction to Theorem 1.

**Theorem 4.** Let \( \Omega = \{ x: x' \in \Omega', -\infty < x_n < +\infty \} \), \( \Delta u = f \) in \( \Omega \), \( u|_{\partial \Omega} = 0 \), \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \), \( \partial \Omega \in C^1 \).

Then \( u \equiv 0 \) in \( \Omega \) provided there is a sequence \( R_j \to \infty \) satisfying the inequality (10) with \( \lambda \) defined by the relation (11) and \( \varepsilon(R_j) \to 0 \) for \( R_j \to \infty \).

Similarly as in the case of the Neumann problem the constant

\[
2 \lambda^2 \text{ in the inequality (10) is the best possible which is demonstrated by the example of the solution } u(x) = w(x') \exp \left\{ \lambda^2 x_n \right\}
\]

of the equation \( \Delta u = 0 \) where \( w(x') \) is the eigenfunction corresponding to the first eigenvalue of the Dirichlet problem

\[
\sum_{j=1}^{n-1} w_{x_j} x_j + \lambda w = 0 \text{ in } \Omega', \quad w|_{\partial \Omega'} = 0.
\]

**Theorem 5.** Let \( \Omega \) be a subset of the half-space \( x_n > 0 \) and let \( S_\zeta = \Omega \cap \{ x: x_n = \zeta \} \) be a domain in the plane \( x_n = \zeta \). Let \( \Delta u = f \) in \( \Omega \), \( u|_{\partial \Omega} = \psi \) with \( f \equiv 0 \) and \( \psi \equiv 0 \) for \( x_n \leq T \), \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \). Then
(12) \[ \int_{\Omega} |\nabla u|^2 dx \leq \exp \left\{ - \int_{\Omega_0} \frac{1}{2} \lambda^2(z) dz \right\} \int_{\Omega} |\nabla u|^2 dx, \]

where

\[ \Omega_z = \Omega \cap \{ x: x_n < z \}, \quad \Omega_0 < \Omega_1 \leq \Omega, \]

\[ \lambda(z) = \inf_{v \in \mathcal{N}_z} \left\{ \int_{S_z} \sum_{j=1}^{N-1} \frac{v^2}{x_j^2} dx' \left[ \int_{S_z} v^2 dx' \right]^{-1} \right\}, \]

\( \mathcal{N}_z \) is the family of all functions \( v(x') \) continuously differentiable in \( \overline{S_z} \) satisfying \( v = 0 \) on \( \partial \Omega \cap \overline{S_z} \).

In this way the exponential factor occurring in the inequality (12) analogous to the Saint-Venant principle, can have an arbitrary character of decrease depending on the metric properties of the domain. Generalizations of Theorems 1 to 5 to the case of elliptic and parabolic equations of the second order in domains \( \Omega \) of general shapes are given in [5] - [7].

Let us now consider the Saint-Venant principle for the biharmonic equation which results from plane problems of the linear elasticity theory. Let \( \Omega \) be a bounded domain in the plane \((x_1, x_2)\) from the class \( C^1 \) such that \( \Omega \subset \{ x: x_2 > 0 \} \) and the intersection of the domain \( \Omega \) with the straight line \( x_2 = z \) is a set \( S_z \) which consists of a finite number of intervals. Let \( l(z) \) equal the length of the largest interval from \( S_z \), \( z = \) const \( > 0 \). In the domain \( \Omega \) let us consider the equation

(13) \[ \Delta^2 u = f, \quad \Delta^2 u = u_{x_1 x_1 x_1 x_1} + 2u_{x_1 x_1 x_2 x_2} + u_{x_2 x_2 x_2 x_2}, \]

with boundary conditions

(14) \[ u|_{\partial \Omega} = \psi_1, \quad \frac{\partial u}{\partial \nu}|_{\partial \Omega} = \psi_2, \]

assuming that \( f \equiv 0 \) in \( \Omega_T \), \( \psi_1 \equiv 0 \), \( \psi_2 \equiv 0 \) on \( \partial \Omega \cap \Omega_T \) where \( \Omega_T = \Omega \cap \{ x: x_2 < z \} \), \( \nu \) is the direction of the outer normal to \( \partial \Omega \), \( T = \) const \( > 0 \). In the domain \( \Omega \) we obtain an estimate for \( u(x) \) which expresses the Saint-Venant principle for a two-dimensional elastic body. Special cases of this estimate are established in [3], [12] in a different way.

**Theorem 6.** Let \( u(x) \) be a solution of the problem (13), (14)
in a domain $\Omega$, $f \equiv 0$ in $\Omega_T$, $\psi_1 = \psi_2 = 0$ on $\partial\Omega \cup \partial\Omega_T$, \( u \in C^4(\Omega) \cap C^3(\Omega^c) \). Then an estimate

\[
\int_{\Omega} E(x)dx \leq \left[ \phi(\varphi_0, \varphi_1) \right]^{-1} \int_{\Omega} E(x)dx
\]

holds, where $E(x) = u_{x_1}^2 + 2u_{x_1}^2x_2 + u_{x_2}^2$, $\varphi_0 < \varphi_1 \leq T$, the function $\phi(x_2, \varphi_1)$ satisfies the identity

\[
\phi_{x_2} - \mu(x_2) \phi = 0
\]

for $\varphi_0 \leq x_2 \leq \varphi_1$ and the initial conditions

\[
\phi(\varphi_1, \varphi_1) = 1, \quad \phi_{x_2}(\varphi_1, \varphi_1) = 0,
\]

where $\mu(\varphi)$ is an arbitrary continuous function satisfying

\[
0 < \mu(\varphi) \leq \lambda(\varphi) = \inf_{v \in \mathbb{N}} \left\{ \int_{S_\varphi} \left[ \int_{S_\varphi} \left( \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_1^2} \right)dx_1 \right]^{-1} \right\},
\]

$N$ is the family of functions $v(x_1, x_2)$ twice continuously differentiable in a neighborhood of $S_\varphi$ and such that $v = 0$, $v_{x_1} = 0$, $v_{x_2} = 0$ at the endpoints of the intervals from $S_\varphi$.

Proof. Integrating by parts we obtain

\[
0 = \int_{\Omega} \phi u \Delta u \, dx = \int_{\Omega} E \phi \, dx - \int_{\Omega} \left( \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_1^2} \right) \phi_{x_2} \, dx_2 + \int_{S_\varphi} \left( u_{x_2} x_2^2 \phi - u_{x_2} x_2 \phi_{x_2} - u_{x_1} x_2 \phi_{x_1} \right) \phi \, dx_1 + \int_{S_\varphi} \left( u^2_{x_2} - uu_{x_2} + u^2_{x_1} \phi \right) \, dx_1.
\]

This implies

\[
\int_{\Omega} E \phi \, dx = \int_{\Omega} \left( \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_1^2} \right) \phi_{x_2} \, dx_2 - \int_{S_\varphi} \left( u^2_{x_2} - uu_{x_2} + u^2_{x_1} \phi \right) \, dx_1.
\]
Taking here \( \phi = 1 \) we conclude

\[
\int_{\Omega} \mathbf{E}(x) dx = - \int_{S_{\tau_1}} \left( u_{x_2} x_2 u - u_{x_2} x_2 - u_{x_1} x_2 u_{x_1} \right) \phi dx_1 - \\
- \int_{S_{\tau_1}} (u_{x_2}^2 - uu_{x_2} x_2 + u_{x_1}^2) \phi x_2 dx_1.
\]

Let us introduce a function \( \phi = \phi(x_2, \tau_1) \) defined for \( \tau_0 \leq x_2 \leq \tau_1 \) by the equation (16) and initial conditions (17) and continued linearly for \( 0 \leq x_2 \leq \tau_0 \) so that for \( x_2 = \tau_0 \) the function \( \phi \) is continuous and has a continuous derivative \( \phi_{x_2} \).

Taking into account (18) we obtain from (19), (20)

\[
\int_{\Omega} \mathbf{E} \phi dx \leq \int_{\Omega} \mathbf{E} \mu^{-1}(x_2) \phi x_2 dx_2 + \int_{\Omega} \mathbf{E} dx.
\]

Hence with regard to (16)

\[
\int_{\Omega} \mathbf{E}(x) \phi(x_2, \tau_1) dx \leq \int_{\Omega} \mathbf{E}(x) dx.
\]

Let us now study the function \( \phi(x_2, \tau_1) \). We shall show that

\( \phi > 0, \phi_{x_2} < 0 \) for \( 0 \leq x_2 < \tau_1 \). Integrating the equation

\[
\phi_{x_2} x_2^2 - \mu(x_2) \phi = 0
\]

from \( x_2 \) to \( \tau_1 \) we obtain

\[
\phi_{x_2}(x_2, \tau_1) = \int_{x_2}^{\tau_1} \mu(x_2) \phi(x_2, \tau_1) dx_2, \quad \tau_0 \leq x_2 \leq \tau_1.
\]

If the inequality \( \phi(x_2, \tau_1) > 0 \) is not valid for \( \tau_0 \leq x_2 \leq \tau_1 \) then there exists a point \( x_2 = \alpha \) such that \( \phi(\alpha, \tau_1) = 0 \), \( \phi(x_2, \tau_1) > 0 \) for \( x_2 > \alpha \). Obviously \( \phi_{x_2}(\alpha, \tau_1) \geq 0 \). On the other hand, the relation (22) implies that \( \phi_{x_2}(\alpha, \tau_1) < 0 \).
The contradiction just obtained proves that $\Phi(x_2, \tau) > 0$, $x_2 \leq \tau$. The identity (22) implies that $\Phi(x_2, \tau) < 0$ for $\tau_0 < x_2 < \tau$. Consequently, the inequality (21) implies the estimate (15). The theorem is proved.

Let us now assume that $\lambda(x_2) \geq \mu = \text{const} > 0$. Then for $\tau_0 \leq x_2 \leq \tau$

$$\Phi(x_2, \tau) = \frac{1}{2} \left[ \exp \left\{ \frac{1}{\mu^2} (\tau - x_2) \right\} + \exp \left\{ -\frac{1}{\mu^2} (\tau - x_2) \right\} \right].$$

In this case the estimate (15) implies the inequality

$$\int_{\tau_0}^{\tau} E(x)dx \leq 2 \exp \left\{ -\frac{1}{\mu^2} (\tau - \tau_0) \right\} \int_{\tau}^{\tau'} E(x)dx.$$

Let us now estimate $\mu$. It is known that if $v(x_1, x_2)$ is such that $v = 0$, $v_{x_1} = 0$, $v_{x_2} = 0$ at the endpoints of the intervals from $S_\tau$, then

$$\int_{S_\tau} v^2dx_1 \leq \lambda_1^{-1}(\tau) \int_{S_\tau} v_{x_2}^2dx_1, \quad \lambda_1 = \frac{\pi^2}{1^2(\tau)},$$

$$\int_{S_\tau} vdx_1 \leq \lambda_2^{-1}(\tau) \int_{S_\tau} v_{x_1}dx_1, \quad \lambda_2 = \frac{(4,73)^4}{1^4(\tau)},$$

$$\int_{S_\tau} v^2dx_1 \leq \lambda_3^{-1}(\tau) \int_{S_\tau} v_{x_1}^2dx_1, \quad \lambda_3 = \frac{4 \pi^2}{1^2(\tau)}.$$

Here $l(\tau)$ is the length of the largest interval from $S_\tau$. Let $l = \sup_{0 \leq \tau \leq T} l(\tau)$. With regard to the above inequalities we obtain

$$\left| \int_{S_\tau} \left( v_{x_2}^2 - vv_{x_2}x_2 + v_{x_1}^2 \right)dx_1 \right| \leq \frac{1}{2} \lambda_1^{-1} \int_{S_\tau} 2v_{x_1}x_2dx_1 +$$

$$+ \lambda_2^{-1} \int_{S_\tau} v_{x_1}dx_1 + \frac{\theta}{2} \int_{S_\tau} v_{x_2}^2dx_1 + \frac{\lambda_3^{-1}}{2} \int_{S_\tau} v_{x_1}^2dx_1,$$

where $\theta = \text{const} > 0$. Let us choose $\theta$ so that $\frac{1}{2} \theta = \lambda_3^{-1} + \frac{1}{2}(\theta \lambda_2)^{-1}$. By an easy computation we have $\frac{1}{2} \lambda_1^{-1} > \lambda_3^{-1} + \frac{1}{2}(\theta \lambda_2)^{-1} = \frac{\theta}{2}$. Therefore $\lambda(\tau) \geq 2 \lambda_1(\tau) \geq 2 \frac{\pi^2}{1^2} = \mu$. The estimate (23) implies the inequality
This estimate is better than the corresponding ones obtained in [2], [12]. The following theorem is analogous to the Phragmen-Lindelöf theorem for the biharmonic equation.

**Theorem 7.** Let \( \Omega \subset \{x: x_2 > 0\} \), let the set \( S_\tau = \Omega \cap \{x: x_2 = \tau\} \) be nonempty for all \( \tau > 0 \), \( f \equiv 0 \) in \( \Omega \), \( \varphi_1 \equiv 0, \varphi_2 \equiv 0 \) on \( \partial \Omega \). Let \( u(x) \) be a solution of the problem (13), (14) and \( u \in C^4(\Omega) \cap C^3(\overline{\Omega}) \). Then \( u \equiv 0 \) in \( \Omega \) provided there is a sequence of numbers \( R_j \to \infty \) for \( j \to \infty \) and a constant \( d > 0 \) such that

\[
\int_{\Omega_{R_j}} E(x)dx \leq \varepsilon(R_j) \left[ \phi(d, R_j) \right]^{-1},
\]

where \( \varepsilon(R_j) \to \infty \) for \( R_j \to \infty \).

**Proof.** By virtue of Theorem 6 and the condition (24) we have

\[
\int_{\Omega_{d}} E(x)dx \leq \left[ \phi(d, R_j) \right]^{-1} \int_{\Omega_{R_j}} E(x)dx \leq \varepsilon(R_j)
\]

for any \( R_j \). Hence

\[
\int_{\Omega_{d}} E(x)dx = 0
\]

and consequently, \( u \equiv 0 \) in \( \Omega_{d} \) since \( u = 0, u_{x_1} = 0, u_{x_2} = 0 \) on \( \partial \Omega \). It is known that a solution of the equation \( \Delta^2 u = 0 \) is an analytic function in \( \Omega \). Hence \( u \equiv 0 \) in \( \Omega \).

Given an unbounded domain \( \Omega \) such that \( \lambda(\tau) \geq \mu = \text{const} > 0 \), the condition (24) can be written in the form

\[
\int_{\Omega_{R_j}} E(x)dx \leq \varepsilon(R_j) \exp \left\{ \frac{1}{2} \mu R_j \right\}.
\]

The problem whether the constant \( \mu^{\frac{1}{2}} \) in the condition (25) is the best possible remains open. Theorems analogous to Theorem 6 and 7 can be established in the same way also for more complicated domains \( \Omega \), in particular, for the case of a domain \( \Omega \) which has several branches which stretch to infinity along various directions. Such domains are studied for elliptic equations of the second
order in [5], [6]. The method used here for investigating the prob-
lems (13), (14) was former used in [10] to study the behavior of
solutions of the system of equations of the elasticity theory at
non-regular points of the boundary. Analogous results may be esta-
blished also for solutions of the problem (13), (14). In particular,
the following theorem holds.

**Theorem 8.** Let a bounded domain $\mathcal{Q}$ belong to the halfplane
$\{x: x_2 > 0\}$, $\mathcal{C} = \mathcal{Q} \cap \{x: x_2 = 0\}$ being nonempty. Let $u(x)$ be
a solution of the problem (13), (14), $u \in H_2(\mathcal{Q}) \cap \mathcal{O}^4(\mathcal{Q}) \cap
\cap \mathcal{O}^3(\mathcal{Q} \setminus \mathcal{C})$ and let the curve $\gamma \mathcal{Q} \setminus \mathcal{C}$ belong to the class
$\mathcal{C}^1$, $\gamma \equiv 0$, $\psi_1 \equiv 0$, $\psi_2 \equiv 0$ in a certain neighborhood of the
set $\mathcal{C}$. Then

$$\int_{\mathcal{Q}} E(x) \phi(x_2) dx < \infty,$$

where $\phi(x_2)$ satisfies the equation $\frac{1}{2} \mu(x_2) \phi = 0$ and
the initial conditions $\phi(\alpha) = 1$, $\phi(x_2(\alpha)) = 0$, $0 < x_2 \leq \alpha$
where $\alpha$ is a constant, the function $\mu(x_2)$ is defined by the
relation (18) and by the assumption $\mu(x_2) \to \infty$ for $x_2 \to 0$.

It is possible to establish estimates for the function $\phi(x_2)$
which characterize the growth of $\phi(x_2)$ for $x_2 \to 0$ in dependence
on the geometric properties of the domain $\mathcal{Q}$ in a neighborhood
of the set $\mathcal{C}$.

Let us remark that estimates analogous to the Saint-Venant prin-
ciple for solutions of the Dirichlet problem for the system of equa-
tions of the elasticity theory are established in [10] while for
the mixed problem they are given in [11]. Inequalities analogous to
the Saint-Venant principle as well as theorems of Phragmen-Lindelöf
type which are their consequences, hold under certain conditions for
solutions of general boundary value problems for both elliptic and
parabolic equations. These estimates are given in [13] – [15]. In
these papers an approach is used which is connected with a study of
analytic continuation of solutions in a domain of variation of one
of the independent variables of some specially constructed auxiliary
systems.

**References**

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