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SOLUTION OF SYMMETRIC POSITIVE SYSTEMS OF DIFFERENTIAL EQUATIONS

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S.K.Godunov and the author showed in 1969 that the hyperbolic system obtained from the one-velocity kinetic equation in P_{2n+1} - approximations of the method of spherical harmonics under boundary conditions of Vladimirov's type is symmetric positive. Writing the system of equations of the method of spherical harmonics in the form of a symmetric system in the sense of Friedrichs together with a proof of dissipativity of the boundary conditions have made it possible to discover new qualitative laws of the theory of spherical harmonics. Under general assumptions concerning the dissipation indicatrix, the author proved weak convergence. A little later V.Skoblikov and A.Akišev studied the problem of strong convergence of the method of spherical harmonics. It is also important that the symmetry of the system and the positivity of the boundary conditions allowed to construct effective computing algorithms for the solution of the three-dimensional system of equations of spherical harmonics. The present paper offers a survey of results of the study of symmetric positive systems which appear in the method of spherical harmonics.

1. Formulating the problem

Let G be a convex domain in the three-dimensional Euclidean space R_3 whose boundary is a smooth surface Γ. Let us assume that the surface Γ belongs to the class C^1 and has a bounded radius of curvature at any point. In the cylindric domain $S_T = [0,T]\times G\times\Omega$ with the base $Q = G\times\Omega$ we consider the following initial-boundary value problem for non-stationary one-velocity kinetic transport equation:

(1) $$Lu \equiv \frac{\partial u}{\partial t} + \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \sigma u - \frac{\sigma_s}{4\pi}\int_\Omega g(\bar\omega,\bar\omega')u\,d\bar\omega' = f ,$$

(2) $$u(0,\bar r,\bar\omega) = \Phi(\bar r,\bar\omega) ,$$

(3) $$u(t,r,\omega) = 0 \quad \text{for} \quad (\bar\omega,\bar n)\le 0, \quad \bar r\in\Gamma$$

where $\bar r = (x,y,z)$ are space coordinates, $\omega = (\xi,\eta,\zeta)$ are angular variables which may be written in terms of spherical coordinates as $\xi = \cos\varphi \sin\theta$, $\eta = \sin\varphi \sin\theta$, $\zeta = \cos\theta$, $\Omega = \{\varphi, \theta : 0 \le \varphi \le 2\pi \ , 0 \le \theta \le \pi\}$, $u(t,\bar{r},\bar{\omega})$ is the function of distribution of particles, $f(t,\bar{r},\bar{\omega})$ is the source function, $\sigma(\bar{r})$, $\sigma_s(\bar{r})$ are the macroscopic sections characterizing the properties of the medium, $\frac{1}{4\pi} g(\bar{\omega},\bar{\omega}')$ is the dissipation indicatrix, $\bar{\omega} \in \Omega$, $\bar{\omega}' \in \Omega'$.

Definition 1. A function $u(t,\bar{r},\bar{\omega}) \in C([0,T]; L_2(G \times \Omega))$ is called a generalized solution of the problem (1)-(2) if it satisfies the integral identity

$$(4) \qquad \int_{S_T} L^* vu\,dt\,d\bar{r}\,d\bar{\omega} + \int_{t=0} \Phi(\bar{r},\omega) v(0,\bar{r},\omega)\,d\bar{r}\,d\bar{\omega} = \int_{S_T} vf\,dt\,d\bar{r}\,d\bar{\omega}$$

for all continuously differentiable $v(t,\bar{r},\bar{\omega})$ with

$$(5) \qquad v(T,\bar{r},\bar{\omega}) = 0 ,$$

$$(6) \qquad v(t,\bar{r},\bar{\omega}) = 0 \ \text{ for } \ (\bar{n},\bar{\omega}) \ge 0, \quad \bar{r} \in \Gamma .$$

The formula (4) which is the basis of the above definition requires that the coefficients of the equation (1) and the initial data fulfil the following minimal conditions:

(a) σ , σ_s are measurable in G and σ, $\sigma_s \in L_2(G)$,

(b) $f \in C([0,T] ; L_2(G \times \Omega))$,

(c) $\Phi \in L_2(G \times \Omega)$.

It is easy to see that the integrals in (4) exist provided the conditions (a)-(c) are fulfilled.

Theorem 1. Let us assume that

(i) the coefficients $\sigma(\bar{r})$, $\sigma_s(\bar{r})$ are bounded and fulfil the Lipschitz condition with respect to x,y,z with constants σ_x , σ_y , σ_z , σ_{sx} , σ_{sy} , σ_{sz} , respectively (e.g. the Lipschitz condition in x:

$$|\sigma(x',y,z) - \sigma(x'',y,z)| \le \sigma_x |x' - x''|);$$

(ii) the source function $f(t,\bar{r},\bar{\omega})$ is bounded, i.e. $|f(t,\bar{r},\bar{\omega})| \le f_0$ and it fulfils the Lipschitz condition with respect to t and x,y,z with constants f_t ,f_x ,f_y ,f_z , respectively;

(iii) the initial function $\Phi(\bar{r},\bar{\omega})$ is bounded, i.e. $|\Phi(\bar{r},\bar{\omega})| \le \Phi_0$ and it fulfils the Lipschitz condition with respect to x,y,z with constants Φ_x , Φ_y , Φ_z , respectively.

Then there exists a bounded generalized solution $u(t,\bar{r},\bar{\omega})$ which is absolutely continuous with respect to t,x,y,z and, moreover, Lipschitz continuous in the domain S_T with respect to t,x,y,z .

A proof requires only to establish a priori estimates for the differences $u(t',\bar{r},\omega) - u(t'',\bar{r},\omega)$, $u(t,\bar{r}',\omega) - u(t,\bar{r}'',\omega)$ by virtue of the integral equation

$$u(t,\bar{r},\omega) = e^{-\int_{t^*}^{t}\sigma(\bar{r}-\omega(t-s))ds} u(t^*,\bar{r}^*,\omega) + \int_{t^*}^{t} e^{-\int_s^t \sigma(\bar{r}-\omega(t-s'))ds'} Bu(s,\bar{r}-\omega(t-s),\omega)ds. \tag{7}$$

Here t^*, $\bar{r}^*$ correspond to the point of intersection of the characteristic $\bar{r} - \omega t = \mathrm{const}$ with the side boundary of the domain S_T and with the plane $t=0$, respectively,

$$u(t^*,\bar{r}^*,\omega) = \begin{cases} 0 & \text{if } t^k > 0 \\ \Phi(\bar{r},\omega) & \text{if } t^k = 0 \end{cases},$$

$$Bu = \frac{\sigma_s}{4\pi}\int_{\Omega} g(\bar{\omega},\bar{\omega}')u(t,\bar{r},\bar{\omega}')d\bar{\omega}' + f .$$

We have

$$|u(t',\bar{r},\bar{\omega}) - u(t'',\bar{r},\bar{\omega})| \le M_1|t'-t''| ,$$
$$d(\bar{r}',\bar{r}'',\bar{\omega})|u(t,\bar{r}',\bar{\omega}) - u(t,\bar{r}'',\bar{\omega})| \le M_2|\bar{r}'-\bar{r}''|$$

where M_i depend on T and the constants which appear in the assumptions of Theorem 1, $d(\bar{r}',\bar{r}'',\bar{\omega}) = \min\{d(\bar{r}',\bar{\omega}), d(\bar{r}'',\bar{\omega})\}$, $d(\bar{r},\bar{\omega})$ is the distance from the point whose coordinates are $\bar{r} = (x,y,z)$ to the boundary Γ along the direction $\bar{\omega}$. A detailed proof of these estimates is to be found in [6] , [7] .

2. Method of spherical harmonics

Let us introduce a projection operator

$$\mathcal{P}_n u = \frac{1}{2\pi}\sum_{k=0}^{n}\sum_{m=0}^{k}\left[C_k^{(m)}(u,C_k^{(m)}) + S_k^{(m)}(u,S_k^{(m)})\right] ,$$

where

$$C_k^{(m)} = (2k+1)\frac{(k-m)!}{(k+m)!}P_k^{(m)}(\zeta)\cos m\varphi ,$$

$$S_k^{(m)} = (2k+1)\frac{(k-m)!}{(k+m)!}P_k^{(m)}(\zeta)\sin m\varphi ,$$

$$(u,v) = \int_0^{\pi}\int_0^{2\pi} uv\sin\theta d\theta d\varphi .$$

Then using the method of spherical harmonics, we determine the approximate solution

$$v_n = \frac{1}{2\pi} \sum_{k=0}^{n} \sum_{m=0}^{k} \left[C_k^{(m)} \varphi_k^{(m)} + S_k^{(m)} \psi_k^{(m)} \right]$$

from the equation

$$(8) \qquad \mathcal{P}_n L v_n = \mathcal{P}_n f \quad .$$

This system of equations together with the corresponding initial and boundary conditions can be written in the form

$$(9) \qquad \frac{\partial v_n}{\partial t} + \frac{\partial [\xi] v_n}{\partial x} + \frac{\partial [\eta] v_n}{\partial y} + \frac{\partial [\zeta] v_n}{\partial z} + \sigma v_n -$$

$$- \frac{\sigma_s}{4\pi} \int_{\Omega} g_n v_n d\omega' = f_n \; ,$$

$$(10) \qquad v_n \big|_{t=0} = \Phi_n(\bar{r}, \bar{\omega}) \; ,$$

$$(11) \qquad v_n \in \mathcal{N}_+^n(s) \; , \quad s \in \Gamma$$

where

$$\mathcal{N}_+^n(s) = \left\{ v_n : (v_n , (n_x \xi + n_y \eta + n_z \zeta) v_n) \geq 0 \right\} ,$$

n_x , n_y , n_z are the components of the outer normal n ; ξ, η, ζ are operators which map a harmonic polynomial v_n onto another such polynomial without increasing its degree [2] .

It is known that the system of equations of the method of spherical harmonics may be written in another form which enables us to express the system in the form of a symmetric hyperbolic system in the sense of Friedrichs, namely

$$(12) \qquad B \frac{\partial v_n}{\partial t} + A_1 \frac{\partial v_n}{\partial t} + A_2 \frac{\partial v_n}{\partial y} + A_3 \frac{\partial v_n}{\partial z} + D v_n = F_n \; ,$$

$$(13) \qquad v_n(0, \bar{r}) = \Phi_n \; ,$$

$$(14) \qquad M v_n(t, \bar{r}) = 0 \qquad \text{for } r \in \Gamma$$

where $v_n = \left\{ \varphi_n^{(m)}, \psi_n^{(m)} \right\}$, B is a positive definite matrix, A_i are symmetric matrices [2] , M is a rectangular matrix satisfying boundary conditions of the type of Vladimirov-Maršak. The boundary matrix $A = n_x A_1 + n_y A_2 + n_z A_3$ has a constant rank on the boundary Γ . In virtue of dissipativity of the boundary conditions (11) it is possible to establish a priori estimates

$$\max_t \int_G \int_{\Omega} v_n^2 d\bar{r} d\omega \leq M_3 \quad ,$$

$$\max_t \int_\Gamma \int_\Omega (\omega, n) v_n^2 ds d\bar{\omega} \le M_4$$

where M_3, M_4 are independent of the number of harmonics n. On the basis of the a priori estimates we can prove also the absolute continuity of $v_n(t,\bar{r},\bar{\omega})$ with respect to $t,\bar{r}$ under the conditions (i)-(iii).

Let $u(t,\bar{r},\bar{\omega})$ be a generalized solution of the problem (1) - (3) which is absolutely continuous with respect to $t,\bar{r}$ and let $v_n(t,\bar{r},\bar{\omega})$ be the corresponding generalized solution of the problem (9) - (11). Then the following theorem [4] is valid.

Theorem 2. The sequence $\{v_n(t,\bar{r},\bar{\omega})\}$ approximates $u(t,\bar{r},\bar{\omega})$ with respect to the norm of the space $C([0,T]; L_2(G\times\Omega))$.

In order to prove Theorem 2 we introduce $W_n = u_n - v_n$ where $u_n = \mathcal{P}_n u$. Then the Green formula

$$\begin{aligned}(15)\quad \int_G\int_\Omega W_n^2(t,\bar{r},\bar{\omega}) d\bar{r} d\omega &= \int_G\int_\Omega W_n^2(0,\bar{r},\bar{\omega}) d\bar{r} d\bar{\omega} - \\ &- 2\int_0^t\int_G\int_\Omega \left|\sigma W_n - \frac{\sigma_s}{4\pi}\int_\Omega g_n W_n d\omega\right| W_n dt d\bar{r} d\omega + \\ &+ 2\int_0^t\int_G\int_\Omega (f - f_n + R_n) W_n dt d\bar{r} d\bar{\omega} - \int_0^t\int_\Gamma\int_\Omega (\bar{\omega},\bar{n}) W_n^2 dt ds d\omega ,\end{aligned}$$

holds for the difference W_n with $f_n = \mathcal{P}_n f$, $g_n = \mathcal{P}_n g$ and

$$\begin{aligned}R_n = \frac{1}{2\pi}\Big\{ \sum_{m=0}^{n} \Big[&\frac{1}{2}\Big(-\frac{(n-m+2)!}{(n+m)!}C_{n+1}^{m-1} + \frac{(n-m)!}{(n+m)!}C_{n+1}^{m+1}\Big)\frac{\partial \tilde{\varphi}_n^{(m)}}{\partial x} + \\ &+ \frac{1}{2}\Big(\frac{(n-m+2)!}{(n+m)!}S_{n+1}^{m-1} + \frac{(n-m)!}{(n+m)!}S_{n+1}^{m+1}\Big)\frac{\partial \tilde{\varphi}_n^{(m)}}{\partial y} + \\ &+ \frac{(n-m+1)!}{(n+m)!}C_{n+1}^{m}\frac{\partial \tilde{\varphi}_n^{(m)}}{\partial z}\Big] + \sum_{m=1}^{n}\Big[\frac{1}{2}\Big(-\frac{(n-m+2)!}{(n+m)!}S_{n+1}^{m-1} + \\ &+ \frac{(n-m)!}{(n+m)!}S_{n+1}^{m+1}\Big)\frac{\partial \tilde{\psi}_n^{(m)}}{\partial x} - \frac{1}{2}\Big(\frac{(n-m+2)!}{(n+m)!}C_{n+1}^{m} + \\ &+ \frac{(n-m)!}{(n+m)!}C_{n+1}^{m+1}\Big)\frac{\partial \tilde{\psi}_n^{(m)}}{\partial y} + \frac{(n-m+1)!}{(n+m)!}\frac{\partial \tilde{\psi}_n^{(m)}}{\partial z}\Big]\Big\}\end{aligned}$$

where $\tilde{\varphi}_n^{(m)}$, $\tilde{\psi}_n^{(m)}$ are the coefficients of the representation of u at the last harmonics.

We estimate (15) by

$$\|W_n\|_t^2 \le \|W_n\|_0^2 + k\int_0^t \|W_n\|_\tau^2 d\tau + \|f-f_n+R_n\|^2 t - \int_0^t \int_T \int_\Omega (\bar{n}, \bar{\omega}) W_n^2 dt ds d\omega$$

with

$$\|W_n\|_t^2 = \int_G \int_\Omega W_n^2 d\bar{r} d\omega .$$

In virtue of the properties of harmonic polynomials we obtain the inequality

$$\int_0^t \int_T \int_\Omega (\bar{n}, \bar{\omega}) W_n^2 dt d\bar{r} d\bar{\omega} \le \|u-u_n\|^2_{C([0,T];L_2)} t .$$

Using this inequality, we conclude finally

$$\|u_n - v_n\|^2_{C([0,T];L_2)} \le C_1(T) \{\| \operatorname{grad} |u-u_n| \|^2_{C([0,T];L_2)} + \|u-u_n\|^2_{C([0,T];L_2)}\}$$

which proves the theorem.

3. Estimate of error

Let us consider the stationary problem for the kinetic equation in the case of the plane geometry. Here the boundary value problem for the equations of the method of spherical harmonics in P_{2n+1} - approximations assumes the form

$$[\mu] \frac{\partial v_n}{\partial z} + \sigma v_n = \frac{\sigma_s}{2} \int_{-1}^1 v_n d\mu' + f_n ,$$

$$\int_0^1 P_{2s+1}(\mu) v_n(0,\mu) d\mu = 0 ,$$

$$\int_{-1}^0 P_{2s+1}(\mu) v_n(H,\mu) d\mu = 0 , \quad s=0,1,\dots,n ;$$

where

$$[\mu] v_n = \frac{1}{2} \sum_{k=0}^{n-1} (2k+1) \mu P_k(\mu) \varphi_k + \frac{1}{2} n P_{n-1}(\mu) \varphi_n .$$

If u is the exact solution of the original problem, then $W_n = v_n - u$ satisfies the equation

$$\mu \frac{\partial W_n}{\partial z} + \sigma W_n = \frac{\sigma_s}{2} \int_{-1}^1 W_n d\mu' + \frac{P_{n+1}(\mu)}{P_{n+1}(0)} \Big[\sigma v_n(z,0) - \frac{\sigma_s}{2} \int_{-1}^1 v_n d\mu' - f_n(z,0) \Big] + f - f_n . \tag{16}$$

The function W_n introduced above satisfies the boundary conditions

(17) $W_n(0,\mu) = v_n(0,\mu)$ if $\mu > 0$,

(18) $W_n(H,\mu) = v_n(H,\mu)$ if $\mu < 0$.

Thus we have formulated a boundary value problem (16) - (18) with respect to $W_n(z,\mu)$. By virtue of the maximum principle [5] this yields an estimate for W_n

$$|W_n| \le \max\left\{\max_{z,\mu}\left|\frac{P_{n+1}(\mu)}{P_{n+1}(0)}(Lv_n - f)_{\mu=0} + R_n(f)\right|,\ \max_{\mu}|v_n(0,\mu)|,\ \max_{\mu}|v_n(H,\mu)|\right\}$$

where

$$R_n(f) = f - f_n \ ; \quad Lv_n = \mu\frac{\partial v_n}{\partial z} - \sigma v_n - \frac{\sigma_s}{2}\int_{-1}^{1} v_n d\mu' \ .$$

4. The splitting method for equations of the method of spherical harmonics

Let us now proceed to the problems of construction of the difference schemes for the solution of the problem (12) - (14). To this aim we divide the interval $[0,T]$ into m equal parts with the step $\tau = T/m$ and replace the problem on the interval $[m\tau,(m+1)\tau]$ by a sequence of one-dimensional problems. The convergence of this method was studied in our former papers [8] . Here we show some new approaches to the realisation of symmetric systems on each intermediate step. We consider the one-dimensional problem

(19) $B\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + Du = f$,

(20) $u(m\tau,x) = \Phi$,

(21) $M_1 u = 0$ for $x=0$, $M_2 u = 0$ for $x=H$.

We introduce the spaces

$$\mathcal{N}_+ = \{u : M_1 u = 0\}, \quad \mathcal{N}_- = \{v : M_2 v = 0\}, \quad \mathcal{N}_0 = \{w : Aw = 0\} .$$

In virtue of the dissipativity of the boundary conditions $(Au,u) = 0$, $u \in \mathcal{N}_+$, $(Av,v) \le 0$, $v \in \mathcal{N}_-$. This implies that the spaces $\mathcal{N}_+$, $\mathcal{N}_-$ are orthogonal with respect to the metric $[u,v] = (Au,v)$. Let $\{u\}$, $\{v\}$, $\{w\}$ form orthogonal systems of bases of the spaces $\mathcal{N}_+$, $\mathcal{N}_-$, $\mathcal{N}_0$, respectively.

Denote the matrix formed by these basis vectors by $L = [\bar{u},\bar{v},\bar{w}]$. Then using the mapping $u=LV$ we can rewrite the system (19) in the form

$$\tilde{B}\frac{\partial V}{\partial t} + \Lambda\frac{\partial V}{\partial x} + \tilde{D}V = g$$

where $\tilde{B} = L^{*}BL$, $\Lambda = L^{*}AL$ is a diagonal matrix. The vector V may be written in the form $V = (V^{+},V^{-},V^{0})$ with V^{+},V^{-},V^{0} corresponding to the blocks $\bar{u},\bar{v},\bar{w}$.

The boundary conditions (21) are

$V^{+} = 0$ for $x=0$,

$V^{-} = 0$ for $x=H$.

After transforming the system into the canonical form it is not difficult to find stable difference schemes.

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