Marco Biroli

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HOMOGENIZATION FOR VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES

Marco Biroli
Politecnico di Milano , Milano (Italy)

1. Introduction.

In this lecture we are interested in convergence and estimates in homogenization for variational and quasi-variational inequalities .
Roughly speaking we have a medium with a known microscopical behaviour and we want a medium with a known macroscopical behaviour (homogeneous in the case of classical homogenization) which approach the initial medium .
We state now the problem in the case of 2^o order linear elliptic equations.
Let be $Y = \prod_{i=1}^{N} [0,y_i] \subset R^N$, $a_{ij} \in L^\infty(Y)$, $i,j=1,\dots,N$, such that

$$\sum_{i,j=1}^{N} a_{ij}(y)\, \xi_i\, \xi_j \geq \lambda\, |\xi|^2 \quad \text{a.e. in } Y \;,\quad \lambda > 0,$$

and we extend the a_{ij} by periodicity .
Let Ω be a bounded open set in R^N with smooth boundary and A^ε from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ defined by

$$(1,1) \qquad \langle A^\varepsilon u,v \rangle = \sum_{i,j=1}^{N} \int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{u}{x_j}(x)\, \frac{v}{x_i}(x)\, dx \;,\quad \varepsilon > 0 .$$

We indicate $\mathcal{A}(y) = [a_{ij}(y)]$, $\mathcal{A}^\varepsilon(x) = \left[a_{ij}\left(\frac{x}{\varepsilon}\right)\right]$.
We consider the problem

$$(1,2) \qquad A^\varepsilon u^\varepsilon = f \qquad f \in H^{-1}(\Omega) .$$

We have at least after an extraction of subsequence

$$(1,3) \qquad \underset{\varepsilon \to 0}{\text{w-lim}}\; u^\varepsilon = u^0 \quad \text{in } H_0^1(\Omega)$$

Problems: (1) Is u^0 a solution to a linear 2^oorder elliptic equation

$$(1,4) \qquad A^0 u^0 = f$$

,$\forall f$, where the coefficients $\mathcal{A}^0 = [a_{ij}^0]$ don't depend on the boundary conditions ?

(2) There are some estimates on $\| u^\varepsilon - u^0 \|_{L^\infty}$?

The second question has a relevant interest for Numerical Analysis , why , for highly oscillating coefficients the Numerical Analysis of (1,2) can be very difficult ; if we have some estimates on $\| u^{\varepsilon} - u^0 \|_{L^\infty}$ we can substitute the problem (1,2) by the problem (1,4) .

The answer to problem (1) is affirmative , (1)(8) , and there are also some explicit expression for the coefficients a^0_{ij} of A^0 , which are constants ; the main tool to study the problem (1) is the "energy method" .

The answer to problem (2) is affirmative if $a_{ij} \in C^1(\bar{Y})$ are Y-periodic and $f \in L^r(\Omega)$, $r > N$; in this case we have

$$(1,5) \qquad \| u^{\varepsilon} - u^0 \|_{L^\infty} \leq C \varepsilon^{1/2} .$$

The main tool to study the problem (2) is the " multiple scales " method,(1) . In 2. we give now some results concerning problems (1)(2) for variational and quasi-variational inequalities .

2. Results .

Let $H(y,u,p)$ be a function measurable for $y \in Y$ and continuous in (u,p) such that

$$(2,1) \qquad |H(y,u,p)| \leq K (1 + |u|^2 + |p|^2) ,$$

$$(2,2) \qquad H(y,u,p)u \geq -K (1 + |u|^2) - K_1 |p|^2 \quad , K_1 < \lambda ,$$

$$(2,3) \qquad | H(y,v,p+q) - H(y,u,p) | \leq C_1(M) \; (|p| + |p||q| + |q|^2) + C_2(\eta)$$

for $|u|, |v| \leq M$, $|v-u| \leq \eta$, $0 < C_1(M)$, $C_2(\eta)$ bounded and $\lim_{\eta \to 0} C_2(\eta) = 0$.
We extend $H(y,u,p)$ to $y \quad R^N$ by periodicity and we define

$$(2,4) \qquad H_{\varepsilon}(x,u,p) = H(\frac{x}{\varepsilon},u,p) \cdot$$

Let ψ be a measurable function and

$$(2,5) \qquad K^{\psi} = \{ v \in H^1_0(\Omega) , v \leq \psi \quad \text{a.e. in } \Omega \} .$$

Let $\theta^k(y)$ be defined by the problem

$$(2,6) \qquad \begin{aligned} - \operatorname{div}_y \mathcal{A}(y) \operatorname{grad}_y \theta^k(y) &= \operatorname{div}_y \mathcal{A}(y) \operatorname{grad}_y y_k \\ \theta^k(y) & \quad \text{Y-periodic} \end{aligned}$$

and $P^{\varepsilon}(x) = \operatorname{grad}_y \theta^k(\frac{x}{\varepsilon}) + I$.
We have , in $L^2(\Omega)$,

$$\underset{\varepsilon \to 0}{\text{w-lim}}\, H_{\varepsilon}(x,u,p) = H_0(u,p) = \frac{1}{|Y|} \int_Y H(y,u,[\text{grad}_y \vartheta^{\kappa}(y)+I]\, p\,)\, dy .$$

We consider now the variational inequalities

$$(2,7_{\varepsilon}) \quad \begin{cases} \langle A^{\varepsilon} u^{\varepsilon}, v-u^{\varepsilon} \rangle + \int_{\Omega} H_{\varepsilon}(x,u^{\varepsilon}, \text{grad}\, u^{\varepsilon})(v-u^{\varepsilon})\, dx \geq 0, \\ \forall\, v \in K^{\Psi} \cap L^{\infty}(\Omega), \quad u^{\varepsilon} \in K^{\Psi} \cap L^{\infty}(\Omega), \end{cases}$$

$$(2,7_0) \quad \begin{cases} \langle A^0 u^0, v-u^0 \rangle + \int_{\Omega} H_0(u^0, \text{grad}\, u^0)(v-u^0)\, dx \geq 0, \\ \forall\, v \in K^{\Psi} \cap L^{\infty}(\Omega), \quad u \in K^{\Psi} \cap L^{\infty}(\Omega). \end{cases}$$

Suppose now $a_{ij}(y) \in C^1(\bar{Y})$ and Y-periodic then <u>$P^{\varepsilon}(x) \in L^{\infty}(Y)$</u>.

Theorem 1 - <u>Suppose</u> $\Psi \in L^{\infty}(\Omega)$ <u>and</u>

(a_1) Ψ <u>is one sided Holder continuous and</u> $K \cap H_0^{1,\infty}(\Omega) \neq \phi$

<u>or</u>

(a_2) $\Psi \in H^{1,q}(\Omega)$ <u>with</u> $q > 2$.

<u>Let</u> u^{ε} <u>be solutions of</u> $(2,7_{\varepsilon})$; <u>there exists a subsequence</u> $\{u^{\varepsilon'}\}$ <u>such that</u>

$$\underset{\varepsilon' \to 0}{\text{w-lim}}\, u^{\varepsilon'} = u^0 \quad \text{in } H_0^1(\Omega)$$

$$\lim_{\varepsilon' \to 0} \mathcal{A}^{\varepsilon'} \text{grad}\, u^{\varepsilon'}\, \text{grad}\, u^{\varepsilon'} = \mathcal{A}^0\, \text{grad}\, u^0\, \text{grad}\, u^0 \quad \text{in } \mathcal{M}(\Omega)$$

<u>where</u> u^0 <u>is a solution of</u> $(2,7_0)$, (5) (6) .

The result of convergence of Th. 1 can be also easily extended to the case of the quasi-variational inequality of the impulse control , (5) .

We observe that the result of Th. 1 can be extended in the more general framework of G-convergence .

Existence results for variational inequalities like $(2,7_{\varepsilon})(2,7_0)$ are given in (7) for hypothesis (a_1) and in (6) for hypothesis (a_2) .

Let be now H= 0 (linear case) , we have :

Theorem 2 - (A) <u>If</u> $\|A\Psi\|_{L^r} \leq C$, $r > N$, α <u>is the De Giorgi-Nash exponent</u>

$$\|u^{\varepsilon} - u^0\|_{L^{\infty}} \leq C\, \varepsilon^{\frac{\alpha}{N-2+3\alpha}} .$$

(B) <u>If</u> $\Psi \in H^{1,r}(\Omega)$, $r > N$,

$$\|u^{\varepsilon} - u^0\|_{L^{\infty}} \leq C\, \varepsilon^{\frac{\alpha}{2(N-2+3\alpha)}} .$$

(C) <u>If</u> $\Psi \in C^{\gamma}(\bar{\Omega})$, $\gamma \in (0,1)$,

$$\|u^{\varepsilon} - u^0\|_{L^{\infty}} \leq C\, \varepsilon^{\frac{\alpha\gamma}{2(N-2+3\alpha)}} \quad (2)\ (3) .$$

A known function f can be also considered (in the case (A) we suppose $f \in H^{1,r}(\Omega)$, in the cases (B) (C) we suppose $f \in L^{r}(\Omega)$) ,(2) (3) . The case of the quasi-variational inequality of the impulse control can be treated by the result (B) of Th. 2 and the Caffarelli - friedman method and we obtain an estimate as in (B) , (4) .

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