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INTEGRAL REPRESENTATIONS OF BOUNDED HARMONIC FUNCTIONS

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The considerations in this paper are based on the following two theorems.

THEOREM 1 [2] : Let $D := \{z \in \mathbb{C} : |z|<1\}$, $H^b(D) := \{f: D \longrightarrow \mathbb{C}$ bounded, analytic $\}$, and $(x_n) \subset D$ discrete. Then the following statements are equivalent:

(1) There exists a sequence $(c_n) \subset \mathbb{C}$ such that for every $h \in H^b(D)$

$$h(o) = \sum c_n h(x_n).$$

(2) $\sup|h(x_n)| = \sup|h(D)|$ for every $h \in H^b(D)$.

By [3] , it is always possible to choose in (1) $c_n \in \mathbb{R}_+$.

THEOREM 2 [4] : Denote by m the Lebesgue measure on D. Then for every bounded measure μ on D there exists $F \in L^1_+(m)$ such that

(1) $\int h d\mu = \int hF dm$ for every $h \in H^b(D)$.

(2) $\|F\|_1 = \|\mu\|$.

The proofs of these theorems make extensive use of the fact that $H^b(D)$ is a Banach algebra. But replacing "analytic" by "harmonic" the theorems contain statements about a linear space.

The aim of the following is to obtain similar theorems on spaces of harmonic functions in a general context, to be more precise:

Let X be a locally compact space with a countable base, $H \subset \mathcal{C}(X)$ a linear space such that $1 \in H$ and r, m two probability measures on X. Consider the following problem: Find conditions such that there exists $F \in L^1_+(m)$ satisfying

$$\int h dr = \int hF dm \quad \text{for every } h \in H^b := \{h \in H : h \text{ bounded}\}.$$

Let $H^b_o(r) = \{h \in H^b : \int h dr = o\}$ and equip the space $L^\infty(m)$ with the weak topology $\sigma := \sigma(L^\infty(m), L^1(m))$. An application of the theorem of Hahn-Banach yields:

PROPOSITION 3: The following statements are equivalent:

(1) There exists $F \in L^1_+(m)$ such that

$$\int h dr = \int hF dm \quad \text{for every} \quad h \in H^b.$$

(2) $-1 \notin \overline{L^\infty_+(m) + H^b_o(r)}^{\sigma}$.

For an examination of condition (2) of proposition 3 we consider as in [1], [3] the following convex cone $K \supset L^\infty_+(m) + H^b_o(r)$:

$$K := \{u \in L^\infty(m) : \exists\, h \in H,\ h \text{ bounded above, } h \leq u \text{ m-a.e., } \int h dr \geq o\}.$$

PROPOSITION 4: The following statements are equivalent:

(1) $-1 \notin K$.

(2) $r \ll_H m$ (i.e. $h \in H$, h lower bounded, $h \geq o$ m-a.e. $\Longrightarrow \int h dr \geq o$).

Using the method of [1] to prove $K = \overline{K}^{\sigma}$, we obtain finally:

THEOREM 5: Let m be a probability measure on X such that $\inf h(\text{support}(m)) = \inf h(X)$ for every $h \in H$. If there exists a probability measure r on X satisfying

(*) For every compact $K \subset X$ there exists $\alpha_K > o$ such that
$$\sup -h(K) \leq \alpha_K \int h dr \quad \text{for every} \quad h \in H_+$$

then for every bounded measure μ on X and every $\varepsilon > o$ there exists $F \in L^1(m)$ such that

(1) $\int h d\mu = \int hF dm$ for every $h \in H^b$.

(2) $\|\mu\| \leq \|F\|_1 \leq (1 + \varepsilon) \|\mu\|$.

REMARKS: 1) The existence of a measure r satisfying condition (*) is guaranteed if H is a nuclear Fréchet space. Hence theorem 5 can be applied to the space H of solutions of a large class of linear elliptic or parabolic differential equations of second order on $\mathbb{R}^n$.

2) If H satisfies the classical Harnack inequality then theorem 5 holds even for $\varepsilon = 0$. This is especially the case for X = D and $H := \{h : D \longrightarrow \mathbb{R} : h \text{ harmonic}\}$. By taking $\mu = \varepsilon_o$ and $m := \sum 2^{-n} \varepsilon_{x_n}$ we obtain theorem 1. In the same manner, theorem 2 follows.

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