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CONTROLLABILITY OF LINEAR AUTONOMOUS PROCESSES

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1. Preliminaries.

We shall examine different kinds of controllability for a control process represented by a family of ordinary differential equations

$$(A,c) \qquad \dot{x} = A\,x - c$$

depending on a control parameter $c : t \to c(t)$, a function of time t with values $c(t) \in \mathbb{R}^n$ belonging to a set of functions

$$C_\Gamma = \{c \in L_{loc}(\mathbb{R}, \mathbb{R}^n) : \ c(t) \in \Gamma, \ \text{a.e.} \ t > 0\}$$

where Γ is a given non empty subset of $\mathbb{R}^n$.

Further, the real $n \times n$ matrix A is independent of t. □

For each $c \in C_\Gamma$, $v \in \mathbb{R}^n$, x defined by

$$(1.1) \qquad x(t,v,c) = e^{tA}\,[v - \int_0^t e^{-sA}\,c(s)\,ds]$$

is the unique solution of (A,c) such that $x(0,v,c) = v$.

Therefore we shall say that v is transferable into $\chi \in \mathbb{R}^n$ by means of (A,c) if $x(t,v,c,) = \chi$ for some $t > 0$, and we shall say that

$$(1.2) \qquad V(t,A,\Gamma,\chi) = \{\int_0^t e^{-sA}\,c(s)\,ds + e^{-tA}\chi \,;\, c \in C_\Gamma\}$$

is the set of points which are transferable into χ at time t.

Symmetrically

$$(1.3) \qquad W(t,A,\Gamma,\chi) = \{e^{tA}\ [\chi - \int_0^t e^{-sA}\,c(s)\,ds] : c \in C_\Gamma\}$$

is the set of points which are reachable from χ at time t .

We shall write $V(t,A,\Gamma)$, $W(t,A,\Gamma)$ instead of $V(t,A,\Gamma,0)$, $W(t,A,\Gamma,0)$, respectively. □

2. Complete controllability.

Defining (A,c) (or (A,Γ)) as completely controllable when

$$(C_1) \qquad \exists\ t > 0 : V(t,A,\Gamma) = \mathbb{R}^n ,$$

is justified by the fact that, according to (1.2), (1.3), $V(t,A,\Gamma) = \mathbb{R}^n$ is equivalent to $V(t,A,\Gamma,\chi) = W(t,A,\Gamma,\chi) = \mathbb{R}^n$, $\forall \chi \in \mathbb{R}^n$, so that (C_1) means that for every pair $v, w \in \mathbb{R}^n$ there exist $t_{v,w} > 0$, $c_{v,w} \in C_\Gamma$, such that $x(0,v,c_{v,w}) = v$, $x(t_{v,w},v,c_{v,w}) = w$.

From the properties of $V(t,A,\Gamma)$:

$$V(t,A,\rho\Gamma) = \rho V(t,A,\Gamma), \quad \rho \in \mathbb{R}$$

$$V(t,A,\Gamma+\chi) = V(t,A,\Gamma) + \int_0^t e^{-sA}\chi\, ds, \quad \chi \in \mathbb{R}^n$$

$$V(t,A,\Gamma) = \text{co}\, V(t,A,\Gamma)$$

$$\overline{V(t,A,\Gamma)} = \overline{V(t,A,\overline{\text{co}\Gamma})}$$

it follows that (C_1) allows us to replace the set Γ by scalar multiples $\rho\Gamma$, $\rho \neq 0$, by translates $\Gamma + \chi$, and by the (topological) closure $\overline{\text{co}\Gamma}$ of its convex hull $\text{co}\Gamma$. □

Further, let us denote by C_Γ^0 the subset of those $c \in C_\Gamma$ which are piecewise constant and let

$$V^0(t,A,\Gamma) = \{\int_0^t e^{-sA} c(s)\, ds : c \in C_\Gamma^0\} .$$

Then we have (A. ANDREINI [1] - A. BACCIOTTI [2])

$$V^0(t,A,\Gamma) = V\ (t,A,\text{co}\Gamma)$$

$$\overline{V^0(t,A,\Gamma)} = \overline{V(t,A,\Gamma)}$$

so that, with respect to (C_1), C_Γ can be replaced by C_Γ^0. □

If (C_1) holds Γ must be unbounded, hence $\mathrm{co}\Gamma$ is the union of half lines (not necessarily lines). □

The dimension of $V(t,A,\Gamma)$, i.e., the dimension of its affine hull, is independent of t and it is $= n$ iff the following condition

$$\text{(a)} \qquad y \in \mathbb{C}^n, \quad A^* y = \lambda y, \quad y^* \Gamma = \text{const.} \Rightarrow y = 0$$

is satisfied.

If (C_1) holds we can assume $0 \in \Gamma = \overline{\mathrm{co}\Gamma}$, so that

$$(2.1) \qquad (C_1) \Rightarrow (a_0)$$

where

$$(a_0) \qquad y \in \mathbb{C}^n, \quad A^* y = \lambda y, \quad y^* \Gamma = 0 \Rightarrow y = 0. \quad □$$

Because of the identity

$$V(t+\tau,A,\Gamma) = V(t,A,\Gamma) + e^{-tA} V(\tau,A,\Gamma), \quad t,\tau > 0$$

if $V(t,A,\Gamma) = \mathbb{R}^n$ then $V(t+\tau,A,\Gamma) = \mathbb{R}^n$ for all $\tau > 0$. Therefore (C_1) gives rise to two possibilities, namely, either

$$(C_1') \qquad V(t,A,\Gamma) = \mathbb{R}^n, \quad \forall\, t > 0$$

(<u>instant</u> complete controllability) or

$$(C_1'') \qquad 0 < \inf\{t > 0 : V(t,A,\Gamma) = \mathbb{R}^n\} < +\infty$$

(<u>delayed</u> complete controllability). □

When Γ is a subspace of $\mathbb{R}^n$ we have

$$V(t,A,\Gamma) = \Gamma + A\Gamma + \ldots + A^{n-1}\Gamma$$

independent of t, and $(C_1) = (C_1') = (a_0)$.

When Γ is a subspace (a_0) is also equivalent to the condition

(b) the orthogonal projection of $\overline{\text{co}\Gamma}$ on every non trivial A^* invariant subspace Y of $\mathbb{R}^n$ ($Y \neq \{0\}$, $Y \supset A^*Y$) contains a line.

In general, (b) $\Rightarrow$ (a_0), but not conversely. In fact we have (R.M. BIANCHINI TIBERIO [3])

$$(2.2) \qquad (C_1') \iff (b)$$

so that condition (b) serves to characterize instant complete controllability. If follows, in particular, that (C_1') requires that $\overline{\text{co}\Gamma}$ contain at least an entire line. □

3. Global controllability.

Let us now denote by $V(A,\Gamma,\chi)$ the set of points which can be transferred into a given point χ at some undetermined time, i.e., let

$$V(A,\Gamma,\chi) = \bigcup_{t>0} V(t,A,\Gamma,\chi) \ .$$

Symmetrically let

$$W(A,\Gamma,\chi) = \bigcup_{t>0} W(t,A,\Gamma,\chi)$$

be the set of points which can be reached from χ. Let also $V(A,\Gamma) = V(A,\Gamma,0)$, $W(A,\Gamma) = W(A,\Gamma,0)$.

A much weaker type of controllability than (C_1) is represented by

$$(C_2) \qquad V(A,\Gamma) = W(A,\Gamma) = \mathbb{R}^n \ .$$

This means that every point v can be transferred into every point w, provided the duration of the transfer is not fixed in advance. So we can say that (A,c) (or (A,Γ)) is globally controllable.

Global controllability does not require that the set Γ be unbounded. □

Actually, (C_2) consists of two properties, namely

(T) $$V(A,\Gamma) = \mathbb{R}^n$$

(<u>global transferability</u> into 0) and

(R) $$W(A,\Gamma) = \mathbb{R}^n$$

(<u>global reachability from</u> 0) which are independent each other. However, since $W(A,\Gamma) = V(-A,-\Gamma)$, one can limit himself to consider (T) or (R). □

With respect to (T) we are allowed to replace Γ by any scalar multiple $\rho\Gamma$, $\rho \neq 0$, or by the convex closure $\overline{\text{co}\Gamma}$, but not by a translate $\Gamma + \chi$. □

If we denote by $\text{lin } V(A,\Gamma)$ the linear hull of $V(A,\Gamma)$, then, obviously (T) $\Rightarrow$ (LV), where

(LV) $$V(A,\Gamma) = \text{lin } V(A,\Gamma). \quad \square$$

Clearly (T) also implies the following property

(C_3) $$0 \in \text{int } V(A,\Gamma)$$

and, actually

(3.1) $$(T) \equiv (LV) \wedge (C_3) .$$

It can be shown that

(3.2) $$0 \in \text{int } V(A,\Gamma) \equiv 0 \in \text{int } W(A,\Gamma)$$

so that if we define

(LW) $$W(A,\Gamma) = \text{lin } W(A,\Gamma)$$

we have

$$(R) \equiv (LW) \wedge (C_3), \qquad (C_2) \equiv (LV) \wedge (LW) \wedge (C_3). \quad \square$$

It can be shown (L.A. KUN [11]) that when Γ is bounded then (T) ((R), (C_2)) holds if and only if (C_3) holds and $\mathrm{Re}\lambda \le 0$ (≥ 0, $=0$), for all the proper values λ of A . □

4. 0-local controllability.

From (3.2) it follows that if (C_3) holds then there is a neighborhood N of 0 such that every point in N can be transferred into every point also in N at some undetermined time. Therefore we can say that (A,c) is 0-locally controllable when (C_3) holds. □

Replacing Γ by $\rho\Gamma$ or by $\overline{\mathrm{co}\Gamma}$ leaves unaltered property (C_3), whereas it can be destroyed by a translation of Γ .

If we introduce the condition

$$\text{(c)} \qquad y \in \mathbb{R}^n,\ A^*y = \lambda y,\ y^*\Gamma \le 0 \ \Rightarrow\ y = 0$$

the following implication holds

$$(C_3) \Rightarrow (a) \wedge (c) .$$

The converse is not true unconditionally. It becomes true, however, if we assume

$$(H_1) \qquad 0 \in \overline{\mathrm{co}\Gamma}$$

(V.I. KOROBOV - A.P. MARINIC - E.N.PODOL'SKII [10]) so that (a) $\wedge$ (c) characterizes (C_3) under the additional assumption (H_1).

This result is the last of a series of steps (S.H. SAPERSTONE - J.A. YORKE [14], S.H. SAPERSTONE [13], R.F. BRAMMER [5], M. HEYMANN - R.J. STERN [7]) aimed at replacing by (H_1) the stronger, classical condition (E.B. LEE - L. MARKUS [12])

$$(H_2) \qquad 0 \in \mathrm{int\ rel\ co}\Gamma$$

where int rel coΓ is the interior of coΓ relative to its affine hull. Such replacement is needed by several applications. □

Recently V.I. KOROBOV [9] gave a different characterization of (C_3) under the assumption (H_1). □

Clearly if

$$(4.1) \qquad \exists\ t > 0 : 0 \in \operatorname{int} V(t,A,\Gamma)$$

then (C_3) follows. The converse is not so obvious, but nevertheless true (R.M. BIANCHINI TIBERIO [3]).

On the other hand if (4.1) holds then there are two possibilities, namely, either

$$(C_3') \qquad 0 \in \operatorname{int} V(t,A,\Gamma), \quad \forall\ t > 0$$

(instant 0-local controllability), or

$$(C_3'') \qquad 0 < \sup \{t > 0 : 0 \notin \operatorname{int} V(t,A,\Gamma)\} < +\infty$$

(delayed 0-local controllability).

Note that, in general

$$0 \le \inf \{t \ge 0 : 0 \in \operatorname{int} V(t,A,\Gamma)\} \le$$
$$\le \sup \{t > 0 : 0 \notin \operatorname{int} V(t,A,\Gamma)\}. \qquad \square$$

It can be shown (D.H. JACOBSON [8]; R.F. BRAMMER [6]; R.M. BIANCHINI TIBERIO [3]) that (C_3') holds if and only if, denoting by con coΓ the conic hull of coΓ, (A, con coΓ) is instantly completely controllable, i.e.,

$$(4.2) \qquad (C_3') \iff V(t,A,\text{con co}\Gamma) = \mathbb{R}^n, \quad \forall\, t > 0. \qquad \square$$

5. Local controllability.

Let us now define the set

$$C(A,\Gamma) = \{x \in \mathbb{R}^n : x \in \operatorname{int} V(A,\Gamma,x)\}.$$

Since, equivalently

$$C(A,\Gamma) = \{x \in \mathbb{R}^n : x \in \text{int } W(A,\Gamma,x)$$

we can say that (A,Γ) is locally controllable if $C(A,\Gamma) \neq \emptyset$, i.e., if

$$(C_4) \qquad \exists \; x \in \mathbb{R}^n : x \in \text{int } V(A,\Gamma,x) \; .$$

This means that there is some $x \in \mathbb{R}^n$, not necessarily $= 0$, and some neighborhood of x whose points can be transferred into each other. Clearly (C_3) means $0 \in C(A,\Gamma)$ and $(C_3) \Rightarrow (C_4)$. □

The main properties of $C(A,\Gamma)$ are (R.M. BIANCHINI TIBERIO [4]):

$$C(A,\Gamma) = C(-A,-\Gamma) = C(A,\overline{\text{co}\Gamma}) = \text{co } C(A,\Gamma) = \text{int } C(A,\Gamma). \qquad □$$

In order to determine those pairs (A,Γ) for which (C_4) holds we have to consider the set

$$R(A,\Gamma) = \{x^0 \in \mathbb{R}^n : A\,x^0 \in \Gamma\}$$

of rest points of (A,c) (M. HEYMANN - R.J. STERN [7]): if $x^0 \in R(A,\Gamma)$ then x^0 is a constant solution of $x = A\,x - A\,x^0$. Then it can be shown (R.M. BIANCHINI TIBERIO [4]) that

$$(5.1) \qquad (C_4) \;=\; (a) \wedge (d)$$

where

$$(d) \qquad R(A, \text{ int rel co}\Gamma) \neq \emptyset \; . \qquad □$$

6. Final remarks.

The relationships among the different kinds od controllability considered here are represented by

$$(C_1') \wedge (C_1'') = (C_1) \Rightarrow (C_2) \Rightarrow (C_3') \wedge (C_3'') = (C_3) \Rightarrow (C_4)$$

an the arrows $\Rightarrow$ cannot be reversed. However when Γ is a cone with vertex at 0 then $(C_1) = (C_4)$ and when Γ is a subspace we have $(C_1') = (C_4)$. □

The set $C(A,\Gamma)$ is the set of locally controllable points. If $x \in C(A,\Gamma)$ then, either $x \in \text{int } V(t,A,\Gamma,x)$, $\forall t > 0$, or $0 < \sup \{t > 0 : x \in \text{int } V(t,A,\Gamma,x)\}$. Therefore $C(A,\Gamma)$ is the union of a subset $C'(A,\Gamma)$ of instant controllability and a subset $C''(A,\Gamma)$ of delayed controllability. It can be seen that if $C(A,\Gamma) \neq \emptyset$ then $C'(A,\Gamma) \neq \emptyset$. □

So far only those pairs (A,Γ) for which (C_1') or (C_3') or (C_4) holds have been characterized, respectively by (2.2), (4.2) and (5.1).

As far as we know similar characterizations of (C_1'') (hence of (C_1)), of (C_2), of (C_3'') (hence of (C_3)) are still lacking. □

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