A DUALITY PRINCIPLE FOR DELAY EQUATIONS

Odo Diekmann
Mathematisch Centrum, Amsterdam, the Netherlands

1. INTRODUCTION

A semigroup associated with a first order differential equation describes how the "state" evolves when time proceeds. Here "state" means any object which singles out a unique solution.

In the theory of functional differential equations with finite delay it is usage to take initial functions (describing past history) for such objects. This note calls attention to an alternative: forcing functions of a special type (the idea is due to Miller [7]). Here we concentrate on the natural duality relation between the two constructions in the linear case. This relation was discovered by Burns & Herdman [1] in a special case of infinite delay. It seems that the result presented here as well as the general validity of the basic idea have not been noticed before. Elsewhere we show that the study of semigroups acting on forcing functions is interesting and useful by itself [4].

2. THE USUAL SEMIGROUP

Let ζ be a given n×n-real-matrix-valued function of bounded variation such that $\zeta(\theta) = 0$ for $\theta \le 0$ and $\zeta(\theta) = \zeta(r)$ for $\theta \ge r$. With the retarded functional differential equation

$$(1) \qquad \dot{x}(t) = \int_0^r d\zeta(\theta)\, x(t-\theta)$$

one can associate a strongly continuous semigroup of bounded linear operators on $C = C([-r,0];\mathbb{R}^n)$ as follows:

$$(2) \qquad (T_1(s)\phi)(\theta) = x_s(\theta;\phi), \qquad -r \le \theta \le 0.$$

Here $x(t;\phi)$ denotes the solution of (1) which satisfies the initial condition

$$(3) \qquad x(\theta) = \phi(\theta), \qquad -r \le \theta \le 0.$$

3. THE UNUSUAL SEMIGROUP

(1) and (3) jointly lead to the Volterra integral equation

$$(4) \qquad x = \zeta * x + f$$

where $f = F\phi$ with

(5) $$(F\phi)(t) = \phi(0) + \int_0^t \int_\tau^r d\zeta(\theta)\phi(\tau-\theta)d\tau.$$

Note that $F\phi$ is constant for $t \geq r$. With (4) one can associate a semigroup of bounded linear operators on several spaces of functions, which are defined on $[0,\infty)$ and which are constant on $[r,\infty)$, as follows:

(6) $$(T_2(s)f)(t) = f_s(t) + (\zeta_t * x)(s)$$

The motivation for this definition stems from the fact that $T_2(s)f$ is precisely the forcing function in the translated equation:

$$x_s = \zeta * x_s + T_2(s)f$$

4. DUALITY

Let $\widetilde{NBV} = \widetilde{NBV}([0,r];\mathbb{R}^n)$ denote the space of bounded variation functions defined on $[0,\infty)$ and such that (i) $f(0) = 0$; (ii) f is constant for $t \geq r$; (iii) f is continuous from the right on $(0,r)$. We norm $\widetilde{NBV}$ by the total variation. Then $\widetilde{NBV}$ realizes the dual space of C, the pairing being given by

(7) $$\langle f,\phi\rangle = \int_0^r df(t)\phi(-t).$$

Let ζ^T denote the transpose of ζ. An explicit representation of solutions and some calculations lead to

<u>Theorem</u>. $T_1(s;\zeta)^* = T_2(s;\zeta^T)$

Lack of space prevents the formulation of the corresponding result for the infinitesimal generators.

5. REMARKS

i) There are (at least) two ways to keep books of the information which fixes the future evolution of a delay system: initial functions and forcing functions. Each way induces a natural semigroup construction which we call T_1 and T_2, respectively. Taking adjoints involves two actions: (i) the structure of the target space $\mathbb{R}^n$ requires changing over to the transposed matrix; (ii) the structure of time-dependence requires changing over from the T_1 point of view to the T_2 point of view or vice versa. This is, we believe, a general principle.

ii) Let ρ denote the reflection: $(\rho\psi)(t) = \psi(-t)$. Then $\langle F\rho\psi,\phi\rangle = (\psi,\phi)$, where $(\cdot,\cdot)$ denotes the well-known bilinear form associated with (1). This observation explains the prominent role of this form in the adjoint theory of [5,6] which ignores the T_2 semigroup construction.

iii) The reflection implicit in (7) is convenient for autonomous equations: one can integrate all equations in the forward direction. In the nonautonomous case one should avoid it and instead integrate the adjoint equation in the backward direction.

iv) With (1) one can associate as well a semigroup on the Hilbert space $M_2 = \mathbb{R}^n \times L_2([-r,0];\mathbb{R}^n)$ by $T_1(s)(\phi^0,\phi^1) = (x(s),x_s)$. The Sobolev space $\tilde{H}_1$ of functions with distributional derivative in L_2 and constant on $[r,\infty)$ realizes the dual space of M_2 (the pairing being defined by $\langle f,(\phi^0,\phi^1)\rangle = f(0)\phi^0 + \int_0^r f'(t)\phi^1(-t)dt$). In this reflexive situation $T_1(s;\zeta)$ and $T_2(s;\zeta^T)$ are adjoints of each other and T_2 is strongly continuous as well. F extends to a mapping of M_2 into $\tilde{H}_1$ [2, Thm. 2.1] and $FT_1(s;\zeta)(\phi^0,\phi^1) = T_2(s;\zeta)F(\phi^0,\phi^1)$. The structural operator studied by Delfour & Manitius [2] is precisely F in disguise. The formulation of their results can be simplified using duality as described above.

v) In (4) one need not to require that ζ is of bounded variation. So the T_2-approach also covers some neutral equations. If ζ vanishes for $t \geq r$, the space of forcing functions with this property is invariant under T_2 and one can restrict T_2 to this space. This amounts to studying certain integral equations. The duality principle still holds in this situation [3].

vi) The T_2-approach has a non-linear analogue and, starting from a variation-of-constants formula, one can build a qualitative theory in this frame-work [4].

6. REFERENCES

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