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Free vibrations for the equation $u_{tt} - u_{xx} + f(u) = 0$ with $f$ sublinear


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Summary: The assumptions on a function $f$ are found under which the equation $u_{tt} - u_{xx} + f(u) = 0$ with the boundary conditions $u(t,0) = u(t,\pi) = 0$ has a nontrivial $2\pi$-periodic solution.

1. Notation.

The symbol $\int v$ denotes the integral of a function $v$ over $(0,2\pi) \times (0,\pi)$. By $L^p_\omega$, $1 \leq p < \infty$ (or $L^\infty_\omega$), we denote the space of real-valued measurable functions $u$ on $\mathbb{R} \times (0,\pi)$, $2\pi$-periodic in the first variable and satisfying $\|u\|_p = (\int |u|^p \omega)^{1/p} < \infty$ (or $\|u\|_\omega = \sup \text{ess}|u(t,x)| < \infty$, respectively).

The functions $e_{jk}$ are defined on $\mathbb{R} \times (0,\pi)$ by

$$e_{jk}(t,x) = \begin{cases} \frac{\sqrt{2}}{\pi} \cos jt \sin kx & \text{for } j, k \in \mathbb{N}, \\ \frac{1}{\pi} \sin kx & \text{for } j = 0, k \in \mathbb{N}, \\ \frac{\sqrt{2}}{\pi} \sin jt \sin kx & \text{for } -j, k \in \mathbb{N}. \end{cases}$$

For $u \in L_1$, we put

$$a_{jk}(u) = \int u e_{jk}.$$ 

2. Weak $2\pi$-periodic solutions of the wave equation.

Let $f$ be a real-valued function on $\mathbb{R}$. A function $u \in L_1$ is called a (weak $2\pi$-periodic) solution to the problem

$$(1) \quad u_{tt} - u_{xx} + f(u) = 0, \quad u(t,0) = u(t,\pi) = 0,$$

if the composed function $f(u)$ belongs to $L_1$ and

$$(j^2 - k^2)a_{jk}(u) = a_{jk}(f(u))$$

for any $j, k$.

In the paper [1] the existence of a nontrivial solution to (1) with $f$ of the form

$$(2) \quad f(u) = |u|^\alpha \text{sgn}(u) \quad (0 < \alpha < 1)$$

is established. In the paper [2] the existence of nontrivial $T$-periodic solutions ($T$ sufficiently large) to (1) is proved for a rather
general class of sublinear functions $f$.

3. Formulation of main results.

Let us denote by $S$ (or $S'$) the set of all functions $f$ which fulfill the following assumptions (S1) - (S4) (or (S1) - (S5), respectively):

(S1) $f \in C(\mathbb{R}, \mathbb{R})$, odd, increasing;

(S2) $f$ is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and $f(u)u \geq f'(u)u^2$ for $u \neq 0$;

(S3) there exist constants $c_1 > 0$ and $\delta \in (0,1)$ such that $f(u) \geq c_1 u^{\delta}$ for $u > 0$;

(S4) there exist constants $c_2, c_3 > 0$ and $p > 2$ such that

$$\int_0^u f(s)ds - \frac{1}{2} uf(u) \geq c_2 |f(u)|^p - c_3 \text{ for } u \in \mathbb{R};$$

(S5) the function $u \rightarrow uf(u)$ is convex.

Let us note that any function $f$ of the form (2) belongs to $S'$ and that $f_1, f_2 \in S'$ and $a, b > 0$ implies $af_1 + bf_2 \in S'$.

**Theorem 1.** For any $f \in S$ there exists a nontrivial solution $u \in L_\infty$ to the problem (1).

**Theorem 2.** Let $f \in S'$ and let us denote $F(u) = \int_0^u f(s)ds$ for $u \in \mathbb{R}$. Then there exists a sequence $\{u_n; n \in \mathbb{N}\}$ of solutions to (1), such that $u_n \in L_\infty$ ($n \in \mathbb{N}$) and $\{ \int_{[F(u_n) - \frac{1}{2} u_n f(u_n)]}; n \in \mathbb{N} \}$ forms a decreasing sequence of positive reals with 0 as a limit point.

4. Sketch of proofs.

a) Let $f \in S$. First we shall seek solutions of the "modified" problem

$$u_{tt} - u_{xx} + f_\varepsilon(u) = 0, \quad u(t,0) = u(t,\pi) = 0,$$

where $f_\varepsilon(u) = f(u) + \varepsilon |u|^{1/p-1} \text{sgn}(u)$ (and $p$ is the same as in (S4)).

b) Approximate solutions for (1) will be obtained as critical points
of functionals $g_{n,\varepsilon}$, defined on $H_n = \text{lin}\{e_{jk} : |j| \leq n, k \leq n\}$ by

$$g_{n,\varepsilon}(u) = -\frac{1}{2} \int (u_x^2 - u_t^2) + \int F_\varepsilon(u),$$

where $F_\varepsilon(u) = \int f_\varepsilon(s)\,ds$.

c) The following assertion plays a fundamental role: For any $a > 0$ there exists $k(a) \in (0,a)$ such that for a sufficiently large $n$ and $\varepsilon \in (0,1)$ there exists a critical point $u_{n,\varepsilon}$ of $g_{n,\varepsilon}$ with $g_{n,\varepsilon}(u_{n,\varepsilon}) = \int (F_\varepsilon(u_{n,\varepsilon}) - \frac{1}{2} u_{n,\varepsilon} f_\varepsilon(u_{n,\varepsilon})) \in [k(a),a]$.

In order to obtain those appropriate approximate solutions, the Ljusternik-Schnirelmann theory is used.

d) Let $\varepsilon \in (0,1)$ be fixed. Then it may be shown (by a monotonicity argument) that a certain subsequence of $\{u_{n,\varepsilon} : n \in \mathbb{N}\}$ converges weakly in $L_p'$ (where $p'$ is conjugate to $p$) to a solution $u_\varepsilon \in L_{p'}$ of (1) and that, moreover, $\int u_\varepsilon f_\varepsilon(u_\varepsilon) \geq 2k(a) > 0$ (i.e. that $u_\varepsilon$ is a nontrivial solution).

e) As $u_\varepsilon$ solves (1), the relation

$$\int_0^1 \left[ f_\varepsilon(u_\varepsilon(t-x,x)) - f_\varepsilon(u_\varepsilon(t+x,x)) \right] dx = 0$$

is valid for a.e. $t$. By using this fact it may be shown that $u_\varepsilon$ belong to $L^\infty$ and are bounded in $L^\infty$ uniformly with respect to $\varepsilon \in (0,1)$.

f) By making use of the above assertion it is possible to obtain by the limiting process for $\varepsilon \to 0$ (again mainly by a monotonicity argument) a solution $u \in L^\infty$ to the problem (1) with $\int u f(u) \geq 2k(a) > 0$, which proves Theorem 1.

g) If $f \in S'$ then it may be shown that the solution $u$ obtained by the above procedure satisfies $\int (F(u) - \frac{1}{2} u f(u)) \in [k(a),a]$, which easily implies the validity of Theorem 2.

References
