Ivan Netuka

Monotone extensions of operators and the first boundary value problem

In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 268--271.

Persistent URL: http://dml.cz/dmlcz/702303

MONOTONE EXTENSIONS OF OPERATORS AND THE FIRST BOUNDARY VALUE PROBLEM

Ivan Netuka
Prague, ČSSR

1. <u>The Keldyš theorem</u>. Let $V \subset \mathbb{R}^m$ be a relatively compact open set and $\mathcal{H}(V)$ be the space of harmonic functions on V. Put $H(V) = \{h \in C(\bar{V});\ h_{|V} \in \mathcal{H}(V)\}$ and $H(\partial V) = H(V)_{|\partial V}$. Thus $f \in H(\partial V)$ if and only if the Dirichlet problem for f has a classical solution. Since $H(\partial V) \neq C(\partial V)$ in general, one is led to the question of a reasonable generalization of the notion of the classical solution.

<u>Definition</u>. The Operator $A: C(\partial V) \longrightarrow \mathcal{H}(V)$ is said to be a <u>Keldyš operator</u> on V, if A is linear, positive and gives the classical solution, provided it exists (i.e. $A(h_{|\partial V}) = h_{|V}$ whenever $h \in H(V)$).

There are constructions producing Keldyš operators (e.g. Perron's or Wiener's method) so that no existence problems arise. On the other hand, the question of uniqueness is far from being evident. One of remarkable results of the classical potential theory reads as follows:

<u>Theorem</u> (M. V. Keldyš, 1941). There is a <u>unique</u> Keldyš operator on V.

(An elementary proof is presented in [5].)

2. <u>Problems</u> (cf. [3]).

P_1: Does the Keldyš theorem extend to other second order linear PDE's of elliptic or parabolic types?

P_2: What can be said about uniqueness, if one considers positive linear (or monotone only) extensions of the classical solution to a larger class of (possibly discontinuous) functions?

P_3: If V is not regular, then $H(\partial V)$ is (as a proper closed subspace) a small (= nowhere dense) subset of $C(\partial V)$. On the other hand, $H(\partial V)$ has to be in a sense large enough to guarantee uniqueness of a Keldyš operator. How to measure the "size" of $H(\partial V)$?

3. <u>Uniqueness of extensions in Riesz spaces</u>. Suppose that B and D are Dedekind complete Riesz spaces, H is a majorizing vector subspace of B and $T: H \longrightarrow D$ is a positive linear mapping. Denote $P_T = \{S;\ S: B \longrightarrow D,\ S \text{ increasing},\ S_{|H} = T\}$, $P_T^o = \{S \in P_T;\ S \text{ linear}\}$, $U_T = \{b \in B;\ S_1(b) = S_2(b),\ S_1, S_2 \in P_T\}$, $U_T^o = \{\ldots \in P_T^o\}$. Clearly, $U_T \subset U_T^o$. In order to characterize these sets of uniqueness, we define for $b \in B$

$\hat{b} = \bigwedge\{h;\ h \geq b,\ h \in H\}$, $\check{b} = \bigvee\{h;\ h \leq b,\ h \in H\}$,
$\hat{T}\, b = \bigwedge\{Th;\ h \geq b,\ h \in H\}$, $\check{T}\, b = \bigvee\{Th;\ h \leq b,\ h \in H\}$.

A Hahn-Banach type argument leads to the following

Proposition. $U_T = U_T^o = \{b \in B;\ \hat{T}\, b = \check{T}\, b\}$.

To give a more convenient description of U_T and U_T^o, put $\hat{H} = \{h_1 \wedge \wedge \ldots \wedge h_n;\ n \in \mathbb{N},\ h_j \in H\}$, and suppose that there is a Riesz subspace $L \subset B$ containing $\bigwedge H_1$ for every bounded set $H_1 \subset \hat{H}$ and that there exists a mapping $T_o : L \longrightarrow D$ having the following properties: (1) $T_{o|H} = T$; (2) T_o is a Riesz homomorphism; (3) $T_o(\bigwedge H_1) = \bigwedge T_o(H_1)$ whenever H_1 is a lower bounded and lower directed subset of $\hat{H}$. With these assumptions we have the following

Theorem. $U_T = U_T^o = \{b \in B;\ T_o(\hat{b} - \check{b}) = 0\}$.

4. Uniqueness of extensions in function spaces. Let us consider a special case. Let Y be a metrizable compact topological space, $B = B(Y)$ be the Riesz space of bounded functions on Y and $H \subset C(Y)$ be a closed vector space linearly separating points of Y and containing a strictly positive function. Recall that the point $y \in Y$ is termed a Choquet point of Y (w.r.t. H), if ε_y (= the Dirac measure at y) is the only positive Radon measure ν on Y satisfying $h(y) = \int h\, d\nu$ for every $h \in H$. The set $Ch_H\, Y$ of Choquet points is of type G_δ. If $L = \{g_1 - g_2;\ g_j$ lower semicontinuous$\}$, then L satisfies hypotheses of Sec. 3. Suppose that D is now a Dedekind complete Riesz space of functions defined on a set V. Let $T_o : L \longrightarrow D$ satisfying (1) and (2) be described by means of a family $M = \{\mu_x;\ x \in V\}$ of Radon measures on Y in the sense that $T_o\, f(x) = \int f\, d\mu_x$ whenever $f \in L$ and $x \in V$. One can prove that the condition (3) holds. (Observe that in view of conditions (1) and (2), M is uniquely determined by T on $\hat{H}$, thus by (3) and a Stone-Weierstrass type argument on $C(Y)$ and, consequently, on L.) A Borel set $Q \subset Y$ is said to be negligible if $\mu_x(Q) = 0$ for every $x \in V$. Given $f \in B(Y)$, denote by $d(f)$ the set of points of discontinuity of f.

Proposition. If $f \in B(Y)$, then $d(f) \subset \{y \in Y;\ \hat{f}(y) \neq \check{f}(y)\} \subset d(f) \cup \cup (Y \setminus Ch_H\, Y)$.

Theorem. The following conditions are equivalent: (i) $C(Y) \subset U_T$; (ii) $C(Y) \subset U_T^o$; (iii) $\{y \in Y;\ \hat{f}(y) \neq \check{f}(y)\}$ is negligible for every $f \in C(Y)$; (iv) $Y \setminus Ch_H Y$ is negligible; (v) $U_T = U_T^o = \{f \in B(Y);\ d(f)$ is negligible$\}$.

5. Applications to PDE's. Problems mentioned in Sec. 2 can be investigated in a natural way in the context of harmonic spaces [1] (cf. [2]) including as examples a wide class of elliptic and parabolic PDE's. Let X be a locally compact space with a countable base. Suppose that with every open set U in X, a linear space $\mathcal{H}(U)$ of real continuous functions on U (called harmonic functions on U) is associated in such a way that $\mathcal{H} = \{\mathcal{H}(U);\ U \subset X \text{ open}\}$ is a sheaf. Then $(X, \mathcal{H})$ is called a harmonic space, if the following axioms hold:

I. The regular sets for the Dirichlet problem form a base of X.

II. If U is open and $\{h_n\}$ is a sequence of functions harmonic on U, $h_n \nearrow h$ and h is locally bounded, then $h \in \mathcal{H}(U)$.

III. $1 \in \mathcal{H}(X)$ and $\mathcal{H}^+(X)$ separates the points of X.

Examples. Let X be a bounded open subset of $\mathbb{R}^m$ and $\mathcal{H}(U) = \{u \in C^2(U);\ \sum_{j=1}^{m} \partial_j^2 u = 0\}$ or $\mathcal{H}(U) = \{u \in C^2(U);\ \sum_{j=1}^{m-1} \partial_j^2 u = \partial_m u\}$.

Consider a relatively compact open subset V of a harmonic space $(X, \mathcal{H})$ and define $H(V)$, $H(\partial V)$ similarly as in Sec. 1. Let T be the operator of the classical solution of the Dirichlet problem (i.e. $T(h_{|\partial V}) = h_{|V}$, $h \in H(V)$). In order to apply results of Sec. 4, put $Y = \partial V$, $D = \mathcal{H}^+(V) - \mathcal{H}^+(V)$ and for $f \in L$ define $T_0 f$ as the Perron type solution for f. Remark that the corresponding family $\{\mu_x;\ x \in V\}$ is then nothing else than the system of harmonic measures. Denote by V_i the set of irregular points of V. It is known (Bliedtner-Hansen) that $\partial V \setminus Ch_{H(\partial V)} \partial V$ is negligible, iff V_i is negligible. Write U, U^o instead of U_T, U_T^o and call a Keldyš set or a K-set, if $C(\partial V) \subset U^o$ or $C(\partial V) \subset U$, respectively.

Answers to questions formulated in Sec. 2 are included in the following theorem. (For further results, details, bibliography, comments and historical remarks, the reader is referred to [4], [5].)

Theorem (Keldyš, Brelot, Lukeš, H. and U. Schirmeiers, Netuka). The following conditions are equivalent:

(1) V is a Keldyš set.

(2) V is a K-set.

(3) For every $f \in C(\partial V)$, $\{y \in \partial V;\ \hat{f}(y) \neq \check{f}(y)\}$ is negligible.

(4) $\partial V \setminus Ch_{H(\partial V)} \partial V$ is negligible.

(5) V_i is negligible.

(6) $U = U^o = \{f \in B(\partial V);\ d(f) \text{ is negligible}\}$.

Remark finally that while the set of irregular points is negligible in the case of elliptic equations for every open set, the same is no longer true e.g. for the heat equation.

References

[1] Constantinescu, C., and A. Cornea: Potential Theory on Harmonic Spaces. Springer 1972.

[2] Hansen, W.: The Dirichlet problem, Equadiff IV, Prague 1977, Proceedings; Lecture Notes in Mathematics 703. Springer 1979, 139 - 144.

[3] Monna, A. F.: Note sur le problème de Dirichlet. Nieuw Arch. 19 (1971), 58 - 64.

[4] Netuka, I.: The classical Dirichlet problem and its generalizations, Potential Theory, Copenhagen 1979, Proceedings; Lecture Notes in Mathematics 787, Springer 1980, 235 - 266.

[5] Netuka, I.: The Dirichlet problem for harmonic functions, Amer. Math. Monthly 87 (1980), 621 - 628.