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THE METHOD OF DISCRETIZATION IN TIME AND PARTIAL DIFFERENTIAL EQUATIONS

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One of the well-known methods of solution of parabolic problems is the Rothe method (or the method of lines). The interval $I = [0,T]$ for the time-variable t is divided into p subintervals I_j of the length $h = T/p$, and at each of the points of division $t_j = jh$, $j = 1,\dots,p$, the derivative $\partial u/\partial t$ is replaced by the corresponding difference quotient $[z_j(x) - z_{j-1}(x)]/h$. In this way, the solution of the given problem is reduced to the solution of p elliptic problems for the functions $z_j(x)$ which are approximations of the required solution $u(x,t)$ at the points t_j. (Here, x is written briefly for $(x_1,\dots,x_N)$.) The function $z_0(x)$ is given by the initial function $u_0(x) = u(x,0)$.

This method has been applied by many authors to the solution of sufficiently general problems (Ladyženskaja, Iljin, etc.). A rather different technics in this method, consisting, in essential, in a new way of treating the corresponding elliptic problems, was developed in my work [1]. It enables to obtain, in a relatively very simple way, apriori estimates needed for proofs of existence and convergence theorems and to get, at the same time, a very good insight into the structure of corresponding solutions. This "improved" Rothe method was called the method of discretization in time. It was followed by other authors (Nečas, Kačur a.o.) and, in particular, it became a base for an extensive study of evolution problems in my seminar at the Technical University in Prague. The method was shown to be applicable to a wide range of evolution problems (to parabolic problems, linear as well as nonlinear, including nontraditional integrodifferential problems and problems with an integral condition, describing complicated processes in the theory of heat conduction, thento hyperbolic problems, problems in rheology, etc.). Numerical as well as theoretical aspects of this method have been examined (convergence questions, including those when elliptic problems, generated by our method, are solved approximately, error estimates with tests of their practical efficiency, existence theorems, regularity properties of the weak, or very weak solutions, etc.). The obtained results are summarized in my new book [2]. With only some exceptions, all these results are published in this book for the first time. I would like to say a few words here about the whole problematics, and, consequently, about the contents of this

book. See also a more extensive surveyable article [3].
To make clear the ideas, let us begin here with a relatively simple parabolic problem

(1) $$\frac{\partial u}{\partial t} + Au = f \text{ in } G \times I,$$

(2) $$u(x,0) = 0,$$

(3) $$B_i u = 0 \text{ on } \Gamma \times (0,T),\ i = 1,\dots,\mu,$$

(4) $$C_i u = 0 \text{ on } \Gamma \times (0,T),\ i = 1,\dots,k-\mu,$$

with A and f independent of t and with homogeneous initial and boundary conditions. Here G is a bounded domain in E_N with a Lipschitz boundary Γ, $f \in L_2(G)$, A is a linear differential operator of order $2k$ with bounded measurable coefficients, (3), or (4) are boundary conditions, stable (thus containing derivatives of orders $\leq k-1$), or unstable, with respect to the operator A, respectively. Applying the above described method, we have to solve, successively for $j = 1,\dots,p$, the equations

$$Az_j + (z_j - z_{j-1})/h = f \text{ in } G,$$

with boundary conditions $B_i z_j = 0$, $C_i z_j = 0$ on Γ (by (3),(4)) and with $z_0 = 0$ by (2). Denote

(5) $$V = \{v;\ v \in W_2^{(k)}(G),\ B_i v = 0 \text{ on } \Gamma \text{ in the sense of traces},\ i = 1,\dots,\mu\}.$$

In the weak formulation, we have to solve the problem of finding successively such functions

(6) $$z_j \in V,\ j = 1,\dots,p$$

(with $z_0 = 0$), which satisfy the integral identities

(7) $$((v,z_j)) + \frac{1}{h}(v, z_j - z_{j-1}) = (v,f)\ \forall\ v \in V.$$

Here $(\cdot,\cdot)$ is the scalar product in $L_2(G)$ and $((\cdot,\cdot))$ is the bilinear form corresponding to the operators A, B_i, C_i, familiar from the theory of variational methods. Let us assume that this form satisfies

(8) $$|((v,u))| \leq K\|v\|_V\|u\|_V,$$

(9) $$((v,v)) \geq \alpha\|v\|_V^2.$$

Then each of the problems (6), (7) is uniquely solvable. Thus it is possible to construct the so-called <u>Rothe function</u> $u_1(x,t)$ - or $u_1(t)$, if considered as an abstract function from I into V - defined in the subintervals I_j, $j = 1,\dots,p$, by

(10) $$u_1(t) = z_{j-1} + \frac{z_j - z_{j-1}}{h}(t - t_{j-1}).$$

Consider the division d_n of the interval I into $2^{n-1}p$ subintervals. Similarly as before, the n-th Rothe function $u_n(t)$ can be constructed. In this way we get the so-called Rothe sequence $\{u_n(t)\}$. Thanks to the just mentioned new technics from [1] one obtains in a simple way the needed apriori estimates. In particular, it turns out that the sequence $\{u_n\}$ is bounded in $L_2(I,V)$ (the space of abstract functions from I into V, square integrable in the Bochner sense) and that, consequently, a subsequence $\{u_{n_k}\}$ can be found, weakly convergent in that space to a function u. It is shown, without difficulties, that this function satisfies

(11) $u \in L_2(I,V) \cap AC(I,L_2(G))$,

(12) $u'_{L_2(I,L_2(G))} \in L_2(I,L_2(G))$,

(13) $u(0) = 0$ in $C(I,L_2(G))$,

(14) $\int_0^T((v,u))dt + \int_0^T(v,u')dt = \int_0^T(v,f)dt \ \forall\ v \in L_2(I,V)$.

Definition. The function u with the properties (11) - (14) is called the weak solution of the problem (1) - (4).

Uniqueness is then easily established, as well as convergence of the whole sequence $\{u_n\}$ to u weakly in $L_2(I,V)$ and strongly in $C(I,L_2(G))$. So we have

Theorem. Let (8), (9) be satisfied. Then there exists exactly one weak solution of the problem (1) - (4) and

(15) $u_n \rightharpoonup u$ in $L_2(I,V)$, $u_n \to u$ in $C(I,L_2(G))$.

Using then the same technics, in [2]

(i) a relatively very sharp error estimate is derived, i.e. estimate of the norm $\|u(t) - u_1(t)\|$ at the points of division t_j;

(ii) convergence of the "Ritz-Rothe method" is proved, i.e. convergence in the case that elliptic problems (6), (7) are solved approximately by the Ritz method, or by a method with similar properties;

(iii) regularity questions are discussed, i.e. smoothness of the weak solution "with respect to x" (see also [4]) as well as "with respect to t".

These results are then extended to the case of nonhomogeneous initial and boundary conditions.

In a similar way, the method of discretization in time is then applied, in [2],

(i) to the case that A and f depend on t,
(ii) to the case of A nonlinear,
(iii) to integrodifferential parabolic problems,
(iv) to parabolic problems with an integral condition,
(v) to linear hyperbolic problems,
(vi) to a problem in rheology.

Summarizing, one can conclude: The method of discretization in time is a powerful numerical method, applicable to the solution of a wide range of evolution problems, while convergence questions can be answered in a relatively very simple way. It produces sufficiently general existence theorems, even in the case of nontraditional problems. Being a very natural method, it permits a new approach to the investigation of properties of the corresponding solutions (a new way of discussing regularity questions, etc.).

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