ON CHAOS IN DIFFERENCE AND DIFFERENTIAL-DIFFERENCE EQUATIONS

Kristína Smítalová
Bratislava, Czechoslovakia

Discrete models of various biological phenomena have usually the form of difference equations, while certain continuous models can be described by differential-difference equations (cf. [2], among others).

It is known that both these types of equations can be chaotic in the sense of Li and Yorke. Moreover, in [3] there is given a simple delay-differential equation whose solutions have chaotic behaviour.

Consider the simple difference equation

(1) $$x_{n+1} = f(x_n)$$

where f is a continuous selfmapping of a compact real interval I. The equation (1) generates for each $x = x_0 \in I$ a sequence x_0, x_1, x_2, ... of points which may have one of the following two types of behaviour: (a) it is assymptotically periodic (in this case the sequence has a finite number of cluster points) or (b) the sequence has infinitely many of cluster points - it is chaotic. When for each x_0 the case (a) occurs then f is non-chaotic. In modelling this represents a highly desirable form of behaviour. However, Cloeden [4] has recently proved the following rather startling result: The chaotic functions are dense in the space $C(I)$ of continuous functions $I \longrightarrow I$ with the max-norm. In [5] we have proved that the chaos cannot be large when a function is near to a suitable non-chaotic function:

<u>Theorem 1</u> (cf. [5]). Let $f: I \longrightarrow I$ be a continuous function. Assume that f has only cycles of order $\leq 2^k$. Moreover, assume that both the set of fixed points of f and the set of cyclic points of f are nowhere dense sets. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each continuous $g: I \longrightarrow I$ with $\|f - g\| < \delta$,

$$\limsup_{n \to \infty} \|f^n - g^n\| < \varepsilon$$

where f^n denotes the n-th iterate of f.

There is a natural connection between the difference equation (1) and a delay-differential equation of the type $x'(t) = h(x(t-1))$. The step method of integration of such an equation generates a sequence of functions. The main aim of this report is to show how the above quoted result can be applied in the continuous case.

First we introduce some notation. Let $\tau > 0$ be a constant, let $\mathcal{S}$ be a system of continuous functions $[0,\tau] \to R$ and ν a continuous bijective mapping $\mathcal{S} \to I$, where I is a compact real interval. Let $\mathcal{F}$ be a system of continuous functions $R^3 \to R$, and assume that for each $\varphi \in \mathcal{S}$, $f \in \mathcal{F}$ there exists a unique solution of the equation

$$(2) \qquad \begin{aligned} x'(t) &= f(t, x(t), x(t-\tau)) & t > \tau \\ x(t) &= \varphi(t) & t \in [0,\tau]. \end{aligned}$$

Denote this solution by $x(t, \varphi, f)$. Finally, if y is a mapping $[0,\infty) \to R$, and $a > 0$, let y_a be as usually the mapping $[0,\tau] \to R$ defined by $y_a(t) = y(t+a)$. Now we are able to give our stability theorem.

<u>Theorem 2</u>. Assume that for each $\varphi \in \mathcal{S}$, $f \in \mathcal{F}$, and each positive integer n, $x_{n\tau}(t, \varphi, f) \in \mathcal{S}$.

(a) Let for some $f \in \mathcal{F}$, and some integer k, the sequence

$$(3) \qquad \nu(\varphi),\ \nu(x_\tau(t,\varphi,f)),\ \ldots,\ \nu(x_{n\tau}(t,\varphi,f)),\ \ldots$$

has for each $\varphi \in \mathcal{S}$ at most k cluster points (it follows that (3) is asymptotically periodic, for each φ). Denote by P the set of those $\varphi \in \mathcal{S}$, for which the sequence (3) is periodic or constant, and assume that $\nu(P)$ is nowhere dense in I. Let $\varepsilon > 0$. Then for each $g \in \mathcal{F}$ sufficiently (uniformly) near to f, the sequence (3) with f replaced by g is for each $\varphi \in \mathcal{S}$ asymptotically periodic up to ε-perturbations. (In other words, there is an asymptotically periodic sequence $z_0, z_1, \ldots$ such that $|z_n - (x_{n\tau}(t,\varphi,f))| < \varepsilon$ for all sufficiently large n.)

(b) If for some fixed $f \in \mathcal{F}$ and $\varphi \in \mathcal{G}$ the sequence (3) is not asymptotically periodic, then for each $g \in \mathcal{F}$ sufficiently near to f there is some $\psi \in \mathcal{G}$ such, that the sequence

$$\nu(\psi),\quad \nu(x_\tau(t, \psi, g)),\ \dots,\ \nu(x_{n\tau}(t, \psi, g)),\ \dots$$

is not asymptotically periodic.

Proof of (a) is based on Theorem 1, part (b) is a consequence of a theorem from [1].

As an illustration of Theorem 2 we give the following

Example. Let $h(a, t)$ be a continuous function $[0, 1]^2 \longrightarrow$ $\longrightarrow [0, 1]$, which is continuously differentiable with respect to the second variable. Moreover, assume that for each a, $h(a, 0) =$ $= \partial h(a, 0)/\partial t = h(a, 1) = \partial h(a, 1)/\partial t = \partial h(a, 1/2)/\partial t = 0$, $h(a, 1/2) = a$, and let $h(a, t)$ for arbitrary fixed a, be monotone in the intervals $[0, 1/2]$ and $[1/2, 1]$. Let $\varphi^a(t) = h(a, t)$, and put $\mathcal{G} = \{\varphi^a;\ a \in [0, 1]\}$. Let $g: [0, 1] \to [0, 1]$ be a continuous function. Let $f_g(t, y)$ be a continuous function $R \times [0, 1] \longrightarrow R$, periodic with period 1, which is defined by

$$f_g(t, \varphi^a(t)) = (\varphi^{g(a)})'(t) \qquad \text{for } t \in [0, 1].$$

Fixe some g and a and consider the equation

$$(4)\qquad \begin{aligned} x'(t) &= f_g(t, x(t-1)) && \text{for } t > 1 \\ x(t) &= \varphi^a(t) && \text{for } t \in [0, 1]. \end{aligned}$$

Clearly for each positive integer n there is some a(n) such that

$$x(t) = \varphi^{a(n)}(t - n) \qquad \text{for } t \in [n, n+1].$$

(In fact, $a(n) = g^n(a)$.) It is easy to verify that when g is chaotic then for suitably choosed a the solution of (4) behaves chaotically. When g has only cycles of finitely many different orders (which are all necessarilly powers of 2) then the sequence of local maxima of each solution of (4) is asymptotically periodic. If moreover the periodic points and fix points of g form a nowhere dense set then the equation is structurally stable, with respect to small change of g, in the sense of Theorem 1.

References

[1] G. J. Butler and G. Pianigiani, Periodic points and chaotic functions in the unit interval, Bull. Austral. Math. Soc. 18 (1978), 255 - 263.

[2] N. Mac Donald, Time lags in biological models, Lecture notes in biomathematics, Springer 1978.

[3] V. An der Heiden, H. O. Walther, Existence of chaos in control systems with delayed feedback, preprint.

[4] P. E. Kloeden, Chaotic difference equations are dense, Bull. Austral. Math. Soc. 15 (1976), 371 - 379.

[5] J. Smítal and K. Smítalová, Structural stability of non-chaotic difference equations, to appear in J. Math. Anal. and Applications.