Jiří Taufer

On the transfer of conditions as applied to solving two-dimensional boundary value problems

In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 339--343.

Persistent URL: http://dml.cz/dmlcz/702317

ON THE TRANSFER OF CONDITIONS AS APPLIED TO SOLVING TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS

Jiří Taufer
Praha, ČSSR

The topic of this lecture was stimulated from two sources:

The first one is an attempt to generalize the method of transfer of conditions from the one-dimensional to the two-dimensional case. The one-dimensional case was studied in [1].

The second source is the S.L. Sobolev's old idea of the closure of numerical processes, see [2].

Let us investigate the usual elimination method from this second point of view in order to show the connection of the transfer of conditions with the direct methods for the solution of algebraic systems arising from boundary value problems.

Let us consider the Dirichlet problem for the Poisson equation $\Delta u = f$ on the rectangle $\langle 0,a \rangle \times \langle 0,b \rangle$ and suppose that we solve it by the finite-difference method. For simplicity let us assume that the boundary conditions are of the form:

$$u(0,y) = u(a,y) = 0 \quad \text{for} \quad 0 \leqq y \leqq b$$

$$u(x,0) = p(x) \quad \text{and} \quad u(x,b) = q(x) \quad \text{for} \quad 0 \leqq x \leqq b.$$

Let us assume that $\frac{a}{b}$ is a rational number and that our algebraic system is the result of using the well-known five-point scheme.

We find:

$$(1) \qquad \begin{bmatrix} A, I, 0, \dots & & \\ I, A, I, \dots & & \\ 0, I, A, I, \dots & & \\ & & \\ \dots\dots\dots 0, I, A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-1} \end{bmatrix} = \begin{bmatrix} h^2 f_1 - u_0 \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{m-1} - u_m \end{bmatrix},$$

where $h = \frac{a}{n} = \frac{b}{m}$

and the vectors u_k and f_k have $n - 1$ components:

$$u_k = \begin{bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{n-1,k} \end{bmatrix}, \qquad f_k = \begin{bmatrix} f_{1k} \\ f_{2k} \\ \vdots \\ f_{n-1,k} \end{bmatrix},$$

where u_{jk} are the approximate values of the solution $u(jh,kh)$ and $f_{jk} = f(jh,kh)$.

The square matrix A of order $n-1$ is of the form:

$$A = \begin{bmatrix} -4, 1, 0, \ldots & & \\ 1, -4, 1, 0 \ldots & & \\ & & \\ \ldots\ldots & & 1, -4 \end{bmatrix}.$$

The vectors u_0 and u_m are given by the boundary conditions.

Now, let us apply the block elimination method to the system (1). We obtain:

$$(2) \quad \begin{bmatrix} A, I, 0, 0, \ldots \\ 0, N_3, N_2, 0, \ldots \\ 0, 0, N_4, N_3, \\ \vdots \\ \ldots \quad 0, N_m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-1} \end{bmatrix} = \begin{bmatrix} h^2f_1 - u_0 \\ N_2h^2f_2 - h^2f_1 + u_0 \\ N_3h^2f_3 - N_2h^2f_2 + h^2f_1 - u_0 \\ \vdots \\ \end{bmatrix},$$

where $N_{k+1} = AN_k - N_{k-1}$ and $N_1 = I$, $N_2 = A$.

Let us consider the k-th row of the system (2):

$$(3) \qquad N_{k+1}u_k + N_ku_{k+1} = h^2 \sum_{i=1}^{k} N_if_i(-1)^{k-i} + (-1)^k u_0$$

and let us examine what will happen when we pass with the parameter h to zero in such a way that $k.h$ = constant. It is clear that the dimensions of the matrices N_k and the vectors u_k tend to infinity in the course of this passage to the limit. The question arises whether the system (3) will be converted into an integral equation of the form:

$$(4) \quad \int_0^a D_1(s,x,y)u(x,y)\,dx + \int_0^a D_2(s,x,y)u_y(x,y)\,dx = d(s,y)$$

for $0 < s < a$, where $k.h = y$.

This idea comes from the following consideration: If the derivative u_y in (4) is replaced by any finite difference, if the integrals are replaced by arbitrary quadrature formulae, and if we require that the equation obtained in this manner be satisfied at the collocation points $s_i = ih$, $i = 1,\ldots,n-1$, than we come to an algebraic system of the form (3).

Our question can be formulated as follows: are there such functions $D_1(s,x,y)$, $D_2(s,x,y)$ and $d(s,y)$ that the system (3) is a difference analog of the integral equation (4)? The answer is negative. However, there exist such functions that the difference analog of the integral equation (4) is algebraically equivalent with the system (3). There exist such regular matrices R_k that the system

$$(5) \qquad R_k\left[N_{k+1}u_k+N_k u_{k+1}\right] = R_k\left[h^2\sum_{i=1}^{k}(-1)^{k-i}N_i f_i + (-1)^k u_0\right]$$

tends in the above sense to the equation (4). One may expect that neither the functions $D_1(s,x,y)$, $D_2(s,x,y)$ and $d(s,y)$ nor the regular matrices R_k are given uniquely.

Now, let us given such a variant of the choice of R_k that the functions $D_1(s,x,y)$ and $D_2(s,x,y)$ can be expressed in terms of elementary functions.

<u>Theorem 1.</u> There exist regular matrices R_k such that the system (5) passes to the following integral equation:

$$(6) \qquad \frac{1}{2}u(s,y) - \frac{1}{2}\int_0^a \frac{\partial}{\partial y}(L(2y,s,x))u(x,y)\,dx + \int_0^a \left[L(2y,s,x) - L(0,s,x)\right]u_y(x,y)\,dx = 2\frac{\partial}{\partial y}\int_0^a L(y,s,x)p(x)\,dx + \int_0^y\left\{\int_0^a \left[L(z+y,s,x)-L(y-z,s,x)\right]f(x,z)\,dx\right\}dz,$$

where

$$L(y,s,x) = \frac{a}{4\pi}\log\left(\frac{1-2e^{-\frac{\pi}{a}y}\cos\left(\frac{\pi}{a}(s+x)\right)+e^{-\frac{2\pi}{a}y}}{1-2e^{-\frac{\pi}{a}y}\cos\left(\frac{\pi}{a}(s-x)\right)+e^{-\frac{2\pi}{a}y}}\right).$$

An immediate consequence of this theorem is the following

<u>Theorem 2.</u> Every solution u of the Poisson equation $\Delta u =$

$= f$ on the rectangle $< 0,a > \times < 0,y >$ with the three conditions

$$(7) \qquad \begin{cases} u(0,z) = u(a,z) = 0 & \text{for} \quad 0 < z < y \\ \text{and} \quad u(x,0) = p(x) & \text{for} \quad 0 < x < a \end{cases}$$

satisfies the integral equation (6) for $0 < s < a$.

This Theorem 2 can be also proved directly, without using the finite-difference method. The above way of establishing Theorem 2 is due to the attempt to show that numerical methods, which arise from the integral equation (6) or (4), are either identical with or similar to direct methods for the solution of algebraic systems arising from the numerical solution of boundary value problems.

The integral equation (6) can be treated in a different way. This possibility is given by Theorem 2. We seek such functions $D_1(s,x,y)$, $D_2(s,x,y)$ and $d(s,y)$ that for every function u which satisfies the Poisson equation on the rectangle $< 0,a > \times < 0,y >$ and the three boundary conditions (7), the identity (4) holds. We say that (4) is the transferred condition of the conditions (7). The (4) means the transferred condition from the part of the boundary onto the line segment $[(0,y),(a,y)]$. Analogously, we can define the transferred condition of a condition given on an arbitrary part of the boundary onto an arbitrary curve. In the discrete case, finding the functions $D_1(s,x,y)$, $D_2(s,x,y)$ and $d(s,y)$ is the forward sweep of the block elimination method. We say that this finding represents the forward sweep also in the continuous case. We treat the equation (4) as an evolution problem for the unknown $u(x,y)$, where the initial condition is given on the line segment determined by $y = b$: $u(x,b) = g(x)$, and find the solution of this problem. In the discrete case it is the backward sweep of the elimination method.

Hence, we have constructed the continuous analog of direct methods for the solution of algebraic systems, arising from the Poisson problem on a rectangle. There is a large number of efficient methods for this problem and my aim was not to study this simple problem but to investigate various applications of the idea of transferring the conditions. In my opinion, the idea of the transfer of conditions can play an important role in the following fields:

1. The study of the existing direct methods for the solution of

algebraic systems arising from boundary value problems and the determination of the strategy for numbering the mesh points.

2. The development of new direct methods.
3. The simplification of boundary value problems, especially by the modification of the domains.
4. The application of the existing routines developed for special regions to more general problems.

In the lecture a few examples were shown to illustrate this application.

References

[1] J. Taufer: Lösung der Randwertprobleme für Systeme von linearen Differentialgleichungen, ACADEMIA, Praha 1973.

[2] S.L'. Sobolev: Nekotorye zamečanija o čislennom rešenii integral'nyh uravnenií. Izv. AN SSSR. Ser. matem. 1956, 20, 413 - 436.