ABOUT THE REPRESENTATION OF SOLUTIONS OF LINEAR ELLIPTIC EQUATIONS OF ARBITRARY ORDER

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1. Let $B_z^R \subset \mathbb{R}^n$ be an (open) ball with center z and g a continuous function defined on S_z^R. The solution of the Dirichlet problem

$$\Delta^m u = 0 \text{ in } B_z^R$$

$$\left.\frac{\partial^j u}{\partial n^j}\right|_{S_z^R} = g_j \quad (j=0,\dots,m-1;\ g_j \in C^{m-j-1}(S_z^R))$$

in a point $x \in B_z^R$ is given by

$$u(x) = \frac{(-1)^{m-1}(R^2-|x-z|^2)^m}{(m-1)!R^{n-1}w_n}\sum_{j=0}^{m-1}\int_{S_z^R}\binom{m-1}{j}g_j(y)\frac{\partial^{m-j-1}}{\partial(R^2)^{m-j-1}}\frac{R^{n-2}}{|x-y|^2}\,d\sigma(y)$$

(see [3]), where w_n denotes the measure of the unit sphere and $d\sigma$ the usual surface-element. Without loss of generality we put $z = 0$. For abbreviation we can write $(S_o^R =: S^R)$

$$u(x) = \sum_{j=0}^{m-1}\int_{S^R} P_j(x,y)\,d\mu_j(y) =: P(\mu_0, \mu_1, \dots, \mu_{m-1}; R)(x) \qquad (1)$$

(supp $\mu_j \subset S^R$). The functions $P_j(x,y)$ are the so-called Poisson kernels for the polyharmonic equation. Let u be a polyharmonic function in B_z^R. Then the following question is of interest. Under which conditions u can be represented by way of a Poisson integral of the form (1)?

On the sphere $S^\varrho (\varrho \le R)$ we introduce spherical coordinates. Let S be the space of the coordinates $(\varphi_1,\dots,\varphi_{n-1})$. For a function f, defined on S^ϱ, we write $f(y) = f(\varphi_1,\dots,\varphi_{n-1},\varrho) = f(\varphi,\varrho)$. Let $W_p^l(S)$ $(1<p<\infty,\ l \ge o$ an integer) be the usual Sobolev-space on S, normed by

$$\|u\|_{p,l} := \left\{\sum_{|\alpha|\le l}\int_S |D^\alpha u(\varphi)|^p\, d\varphi_1\dots d\varphi_{n-1}\right\}^{\frac{1}{p}}.$$

<u>Theorem 1 ([4]):</u> Let u be a solution of the equation $\Delta^m u = g,\ g \in C^\sigma(\overline{B^R})$ $(0<\sigma<1)$, which fulfils the condition

$$\left\| \frac{\partial^j u(\varphi,\varrho)}{\partial \varrho^j} \right\|_{p,1} \leq C < \infty$$

for an arbitrary number ϱ with $R - \varepsilon_0 \leq \varrho < R$ and $j = 0,\dots,m-1$. Then there exist functions $f_j \in W_p^1(S^R)$ $(j=0,\dots,m-1)$, such that

$$u(x) = \int_{B^R} G_R(x,y)g(y)dy + \sum_{j=0}^{m-1} \int_{S^R} P_j(x,y)f_j(y)d\sigma(y).$$

$G_R(x,y)$ is the polyharmonic Green function for the ball. Theorem 1 is a generalization of known results for harmonic functions.

2. We want to generalize the result for general elliptic equations in general smooth domains. Let $L(x,D)$ be an elliptic operator of the order $2m$ and $\Omega \subset \mathbb{R}^n$ a bounded smooth domain. In [6] (see also [3]) it is proved, that under suitable conditions there exist a lot of so-called generalized harmonic measures $\mu^\beta_{x,\alpha}$ $(x \in \bar{\Omega}, \alpha = (\alpha_1,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_n)$ multi-indices of order $|\alpha| = \alpha_1 + \dots + \alpha_n \leq m-1$, $|\beta| = \beta_1 + \dots + \beta_n \leq m-1)$ with supp $\mu^\beta_{x,\alpha} \subset \partial\Omega$, such that any solution $u \in C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ of the equation $Lu = 0$ and the derivatives of the order $\leq m-1$ in Ω can be represented in the form

$$D^\alpha u(x) = \sum_{|\beta| \leq m-1} \int_{\partial\Omega} D^\beta u(y) d\mu^\beta_{x,\alpha}(y) \quad (x \in \bar{\Omega},\ |\alpha| \leq m-1).$$

Comparison with the Poisson formula (1) shows, that in the case of a ball and the polyharmonic equation the harmonic measures are given by the Poisson kernels. Also for more general domains the concrete determination of the harmonic measures is an interesting problem.

Following Berezanskij and Rojtberg (see [2]) we shall consider this question for more general elliptic boundary conditions. Let $L(x,D)$ be a properly elliptic operator. We consider boundary operators $B_j(x,D)(j=1,\dots,m)$ of the order $m_j \leq 2m-1$ and we suppose, that the system (B_j) is normal and covers L (see [3]). We are looking fora solution of the boundary value problem

$$Lu = f \text{ in } \Omega,\ B_j u|_{\partial\Omega} = \varphi_j \quad (j=1,\dots,m). \tag{2}$$

From the general theory follows, that there exists an unique solution, if the coefficients of the operators and the boundary are sufficiently smooth and if do not exist eigensolutions for the given domain. Moreover we suppose $s > \frac{n}{2}$, $N = \{0\}$, where N is the kernel of the boundary value problem (2), and

$$(f, \varphi_1, \dots, \varphi_m) \in W_2^{s-2m}(\Omega) \times \prod_{j=1}^{m} W_2^{s-m_j-\frac{1}{2}}(\partial\Omega).$$

Then there exist functions $\Gamma_x^{(B)}(y) = \Gamma^{(B)}(x,y) \in W_2^{2m-s}(\Omega)$, which are smooth for $x \neq y$, such that the solution of the problem for $x \in \Omega$ has the representation

$$u(x) = (f, \Gamma_x^{(B)}) + \sum_{j=1}^{m} (\varphi_j, C_j' \Gamma_x^{(B)}) \tag{3}$$

(see [2]). The scalar product means the duality between $W_2^{s-2m}(\Omega)$ and $W_2^{2m-s}(\Omega)$ resp between $W_2^{s-m_j-\frac{1}{2}}(\partial\Omega)$ and $W_2^{-s+m_j+\frac{1}{2}}(\partial\Omega)$, C_j' are boundary conditions, which complete the adjoined boundary conditions (B_j') of (B_j) to a Dirichlet system, $\Gamma^{(B)}$ is the Green function associated with the homogeneous boundary conditions $B_j u = 0$. $\Lambda_j(x,y) := \overline{C_j' \Gamma^{(B)}(x,y)}$ are the Poisson kernels of the boundary value problem. If the integer s is sufficiently large, (3) we can write in the following form:

$$u(x) = \int_{\Omega} f(y)\, \Gamma^{(B)}(x,y)dy + \sum_{j=1}^{m} \int_{\partial\Omega} \varphi_j(y)\, \Lambda_j(x,y)d\sigma(y).$$

For the special case of the Dirichlet problem we get a representation of the harmonic measures by smooth functions. Now we consider the following question:
Under which conditions a given solution of the equation $Lu = f$ in a bounded smooth domain Ω can be represented in the form (3) with suitable boundary functions?

In order to formulate a corresponding theorem, we must define the condition (U). The proof of this condition for arbitrary smooth domains and for general elliptic operators seems to be an open problem. Let $\Omega_\varepsilon = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > \varepsilon\}$ $(0 \leq \varepsilon \leq \varepsilon_0)$ and $C(\Omega_\varepsilon)$ be the constant, which appears in the classical Schauder-estimates for the solutions of the boundary value problem (2), proved by Agmon, Douglis and Nirenberg [1]. Then we suppose the condition
(U) : $C(\Omega_\varepsilon) \leq C$ for $0 \leq \varepsilon \leq \varepsilon_0$ and ε_0 sufficiently small.

Theorem 2 (/7/): Let u be a solution of the equation $Lu = f$ in Ω, $f \in W_2^{s-2m}(\Omega)$, $s > \frac{n}{2}$. We suppose further, that the condition (U) and the conditions for the boundary value problem, formulated above, are fulfiled and that

$$\| B_j u \,|\, \partial\Omega_{\varepsilon} \|_{s-m_j-\frac{1}{2}} \leq C$$

holds for $j = 1, \ldots, m$ and $0 < \varepsilon \leq \varepsilon_0$. Then there exist functions $\varphi_j \in W_2^{s-m_j-\frac{1}{2}}(\partial\Omega)$, such that

$$u(x) = (f, \Gamma_x^{(B)}) + \sum_{j=1}^{m} (\varphi_j, \Lambda_j).$$

Using this result, one can prove a maximum inequality, which is sharper than the Agmon-Miranda-inequality in some cases (see /5/).

Literature

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