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Persistent URL: http://dml.cz/dmlcz/702334

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STABILITY AND AVERAGING PROPERTIES OF STOCHASTIC EVOLUTION EQUATIONS

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The theory of averaging for differential equations with quickly oscillating coefficients has been a subject of interest for many authors since early fifties, see e.g. [1] for ODE's, [2],[7] for stochastic differential equations. Recently the results from [7] on averaging in the quadratic mean have been extended to stochastic differential equations in a Hilbert space with unbounded drift terms and applied to stochastic PDE's ([5],[6]). In [5] some stability results are also included. They make it possible to find effective conditions guaranteeing the required averaging properties on an infinite time interval; however, they also may be of some independent interest.

In the present contribution the main results from [5],[6] are summarized. They are restated in a slightly less general, but more transparent form. Consider a parameter-dependent system of semilinear SDE's

\( \frac{dx_\alpha(t)}{dt} = (A_\alpha(t) + f_\alpha(t,x_\alpha(t)))dt + \dot{\Phi}_\alpha(t,x_\alpha(t))dw_t, \quad t \geq t_0, \)

\( x_\alpha(t_0) = \psi_\alpha, \quad \alpha \geq 0, \)

in a real separable Hilbert space \( H \), where \( A : H \rightarrow H \) is an infinitesimal generator of a strongly continuous semigroup \( S_t \), \( w_t \) is a \( K \)-valued Wiener process on \( (\Omega, \mathcal{A}, \mathbb{P}) \) with a nuclear covariance \( W \) (\( K \) - a real separable Hilbert space), \( f_\alpha : \mathbb{R}^+ \times H \rightarrow H \), \( \Phi_\alpha : \mathbb{R}^+ \times H \rightarrow \mathbb{L}(K,H) \) are measurable and satisfy

\[
(2) \quad \| f_\alpha(t,x) - f_\alpha(t,y) \| + \| \Phi_\alpha(t,x) - \Phi_\alpha(t,y) \| \leq \hat{k} \| x - y \|,
\]

\[
(3) \quad \text{lim} \alpha \rightarrow 0^+ \int_{t_1}^{t_2} S_{t_2-s} (f_\alpha(s+t_0,x) - f_0(s+t_0,x)) ds = 0,
\]

\[
(4) \quad \text{lim} \alpha \rightarrow 0^+ \int_{t_1}^{t_2} \text{Tr} \{ (\Phi_\alpha(s+t_0,x) - \Phi_0(s+t_0,x)) W(\Phi_\alpha(s+t_0,x) - \Phi_0(s+t_0,x)) \} ds = 0
\]

for all \( x \in H \), \( 0 \leq t_1 \leq t_2 \), and \( \psi_\alpha \rightarrow \psi_0 \).
Then for any $0 < T < \infty$ we have

\[(5) \lim_{\alpha \to 0^+} \sup_{t \in (t_0, T)} E \|x_\alpha(t) - x_0(t)\|^2 = 0.\]

In the finite-dimensional case it can be seen ([7]) that a similar statement is valid even for $T = +\infty$ provided the limit solution $x_0$ is asymptotically stable. The proof from [7] fails for $\dim H = \infty$, however, in [5] we prove the assertion imposing some restrictions on $S_t$.

**Definition.** A solution $x_0$ of the equation (1) is said to be asymptotically stable in the mean square if

(i) for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \geq 0$ and all solutions $\tilde{x}$ of (1) satisfying $E\|\tilde{x}(t_0) - x_0(t_0)\|^2 < \delta$ we have $E\|x(t) - x_0(t)\|^2 < \epsilon$, $t \geq t_0$,

(ii) there exists $A > 0$ such that for all $\epsilon > 0$, $\delta \in (0, A)$ there exists $T = T(\epsilon, \delta) > 0$ such that for all $t \geq 0$, $\tilde{x}$ satisfying $E\|\tilde{x}(t_0) - x_0(t_0)\|^2 < \delta$ we have $E\|\tilde{x}(t) - x_0(t)\|^2 < \epsilon$, $t \geq t_0 + T$.

**Theorem 2 ([5]).** Let (3), (4) be fulfilled uniformly w.r.t. $t_0 \in \mathbb{R}_+$ and $x \in H$ and assume $S(\cdot) \in C((0, +\infty), \mathcal{L}(H))$, $\varphi_\alpha \to \varphi_0$. Then (5) is valid with $T = +\infty$ provided $x_0$ is asymptotically stable in the mean square and $E\|x_0(t)\|^2$ is bounded for $t \geq t_0$.

In order to obtain effective results on infinite time intervals we still need verifiable criteria for mean-square asymptotic stability. The standard application of Liapunov method leads to some difficulties as the mild solutions of (1) need not possess a stochastic differential. This can be overcome by approximating mild solutions by strong solutions similarly as in [4]. For $v \in \mathcal{C}_{1,2}(\mathbb{R}_+ \times H)$ set

$$\mathcal{L} v(t,x,y) = \frac{\partial}{\partial t} v + \langle v_x(t,x-y), Ax - Ay + f_0(t,x) - f_0(t,y) \rangle + \frac{1}{2} \text{Tr}(\tilde{F}_0(t,x) - \tilde{F}_0(t,y)) v_{xx}(t,x-y)(\tilde{F}_0(t,x) - \tilde{F}_0(t,y))W, \quad (t,x,y) \in \mathbb{R}_+ \times \mathcal{D}(A) \times \mathcal{D}(A).$$

**Proposition 3.** Assume $\mathcal{L} v(t,x,y) \leq \psi(t, v(t,x-y))$, $t \in \mathbb{R}_+$, $x, y \in \mathcal{D}(A)$, where $v \in \mathcal{C}_{1,2}(\mathbb{R}_+ \times H)$ is such that $d_1 \|x\|^2 \leq v(t,x) \leq d_2 \|x\|^2$, $\|v_x\| + \|v_{xx}\| \leq d_3(1 + \|x\|^p)$, $x \in H$, for some $d_1, d_2, d_3, p > 0$ and $\psi : \mathbb{R}_+^2 \to \mathbb{R}$ is measurable, $\psi(t,.)$ is Lipschitzian and concave, $\psi(t,0) = 0$ for all $t \geq 0$. Then all solutions $x_0$ of (1) are asymptotically stable in the mean square provided the trivial solution $x \equiv 0$ of the equation $\dot{x} = \varphi(t,x)$ is asymptotically stable.

**Example.** The stochastic parabolic problem described by...
\begin{align}
\frac{\partial u_\varepsilon}{\partial t} &= \Delta u_\varepsilon + \frac{r_1(t/\varepsilon)u_\varepsilon}{1 + |u_\varepsilon|} + \frac{r_2(t/\varepsilon)u_\varepsilon}{1 + |u_\varepsilon|} \tilde{W}(t,x), \quad t \geq t_0, \quad x \in D \\
(D - \text{a bounded region in } \mathbb{R}^n \text{ with } C_2 \text{ boundary}), \\
\tilde{u}_\varepsilon(0,x) &= u_0(x), \quad \tilde{u}_\varepsilon|_{\partial D} = 0 \text{ can be formally rewritten in the form}
\end{align}

\begin{align}
dx_\varepsilon(t) &= (Ax_\varepsilon(t) + f(t/\varepsilon, x_\varepsilon(t)))dt + \tilde{\Phi}(t/\varepsilon, x_\varepsilon(t))dW_t, \\
x_\varepsilon(t_0^+) = \Phi_\varepsilon,
\end{align}

in the space \( H = L^2(D) \), with \( K = H^k(D) \) - valued Wiener process \( W_t \) \((k > 2n)\), where \( A = \Delta | H^2(D) \cap \mathbb{H}^1(\Omega) \), \( f(t,x)(\theta) = r_1(t)x(\theta) \).

\( (1+|x(\theta)|)^{-1}, \tilde{\Phi}(t,x)h(\theta) = r_2(t)x(\theta)h(\theta)(1+|x(\theta)|)^{-1}, \Theta \in D, h \in K \).

Assume
\begin{align}
\frac{1}{\beta_T} \int_{\beta_T}^{\beta_T+T} r_1(t)dt &\to r_1, \quad \frac{1}{\beta_T} \int_{\beta_T}^{\beta_T+T} (r_2(t)-r_2)^2dt \to 0 \quad T \to \infty, \\
\end{align}

uniformly in \( \beta \geq 0 \) for some \( r_1, r_2 \in \mathbb{R} \), and \(-\lambda_0 + \max(0,r_1) + 1/2\ r_2^2\kappa^2T\omega < 0 \), where \( \lambda_0 > 0 \) is the first eigenvalue of \(-A\) and \( k > 0 \) is such that \( \|\tilde{L}(D)\|_K \leq \kappa \). Then it can be checked that

Theorem 2 and Proposition 3 yield
\begin{align}
\sup_{t \geq t_0} E\|x_\varepsilon(t) - \bar{x}(t)\|^2 &\to 0 \quad \text{as } \varepsilon \to 0, \quad \Phi_\varepsilon \to \bar{\Phi},
\end{align}

where \( \bar{x} \) is the solution of the limit equation
\begin{align}
d\bar{x} = (A\bar{x} + r_1\bar{x}/1+|\bar{x}|)dt + (r_2\bar{x}/1+|\bar{x}|)dW_t, \quad \bar{x}(t_0) = \bar{\Phi} \quad \text{(see [5] for a similar example)}.
\end{align}

Remark. Some extensions of the above results (e.g. averaging in \( L^p(\Omega) \) for \( p \geq 2 \), averaging in probability, statements analogous to Theorems 1, 2 for a cylindrical Wiener process, etc.) can be found in [5],[6].

References


