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THE HELMHOLTZ DECOMPOSITION AND THE WEAK NEUMANN PROBLEM IN L^q

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Throughout this paper by $G \subset \mathbb{R}^N (N \geq 2)$ we denote either a bounded or an exterior domain (that is: $G = \mathbb{R}^N \setminus \Omega$, $\phi \neq \Omega$, $\bar{\Omega}$ compact) with boundary $\partial G \in C^1$. For $R > 0$ let $G_R := G \cap \{ |x| < R \}$ and for $1 < q < \infty$ let

$$\tilde{H}^{1,q}(G) := \{ p : G \longrightarrow \mathbb{R} \mid p \in L^q(G_R) \;\forall\; R > 0,\; \nabla p \in L^q(G_R) \}.$$

Clearly for G bounded this space coincides with the usual Sobolev space $H^{1,q}(G)$, for exterior domains it is much bigger. The quotient space $\hat{H}^{1,q}(G) := \tilde{H}^{1,q}(G)/\mathbb{R}$ equipped with norm $\|\nabla .\|_q$ is a reflexive Banach space (we omit the typografical destinction between p and the equivalence class [p]). Define

$$D_o^\infty(G) := \{\underline{\phi} \in C_o^\infty(G)^N \mid \operatorname{div} \underline{\phi} = 0 \text{ in } G\},\; D^q(G) := \overline{D_o^\infty(G)}^{\|.\|_q}$$

$$G^q(G) := \{ \nabla p \mid p \in \hat{H}^{1,q}(G)\}.$$ Then

<u>Theorem 1</u> (Helmholtz decomposition)

a) $L^q(G)^N = D^q(G) \oplus G^q(G)$ in the sense of an orthogonal decomposition if $q = 2$ and as a direct sum otherwise.

b) There exists a constant $K_q > 0$ such that
$\|\underline{u} + \nabla p\|_q \geq K_q (\|\underline{u}\|_q + \|\nabla p\|_q)$ for $\underline{u} \in D^q(G)$, $\nabla p \in G^q(G)$

c) $D^q(G) := \{ \underline{u} \in L^q(G)^N \mid \langle \underline{u}, \nabla \underline{\Psi} \rangle = 0 \;\forall\; \Psi \in \hat{H}^{1,q'}(G)\}$.

Here for $\underline{f} \in L^q(G)^N$ and $\underline{g} \in L^{q'}(G)^N$ ($q' = \frac{q}{q-1}$) we write $\langle \underline{f}, \underline{g} \rangle := \int_G \sum_{i=1}^N f_i\, g_i dx$.
The Helmholtz decomposition is a basic tool e.g. for the construction of the Stokes operator. Therefore it was studied by many authors: For bounded G and $q = 2$ a proof by means of potential theoretical methods is due to Ladyshenskaya [2]; in this case another proof can be found in the book of Temam [7] using a deeplying theorem due to de Rham. For $q \neq 2$ and bounded G the result is due to Fujiwara-Morimoto [1] using the whole Lions-Magenes theory. For $N = 3$ and $1 < q < \infty$ the first proof for exterior domains was given by Miyakawa [3]. We sketch the main ideas of a very natural approach to the general case considered here. This is a joint work with H. Sohr (Paderborn, West Germany).
For $q = 2$ our proof is completely elementary. It rests on the following

Theorem 2 Let $\underline{u} \in L^q(G)^N$ such that $\langle \underline{u}, \underline{\Psi} \rangle = 0$ for all $\underline{\Psi} \in D_o^\infty(G)$. Then there exists $p \in H^{1,q}(G)$ such that $\nabla p = \underline{u}$.

Sketch of proof: If $\gamma : [0,1] \longrightarrow G$ is an arbitrary closed differentiable curve and $0 < \varepsilon < \text{dist}(\gamma, \partial G)$ and we put $\phi_i^{(\varepsilon)}(x) := \int_o^1 j_\varepsilon(x-\gamma(t))\, \gamma_i'(t)dt$, where j_ε denotes the mollifier kernel, then $\underline{\phi}^{(\varepsilon)} \in D_o^\infty(G)$ and we get $0 = \langle \underline{u}, \underline{\phi}^{(\varepsilon)} \rangle = \int_\gamma \underline{u}_\varepsilon\, d\underline{s}$. But then we conclude for compact subsets G' that $\underline{u}_\varepsilon = \nabla p^{(\varepsilon)}$ with some $p^{(\varepsilon)} \in C^1(G')$. A limiting procedure based on the following Lemma then proves Theorem 2.

Lemma 1. Let $G \subset \mathbb{R}^N$ be a domain and let $(p_n) \subset C^1(G)$, $1 \le q < \infty$. Assume that (∇p_n) is a Cauchy sequence in $L^q_{loc}(G)^N$. Then there is a sequence $(c_n) \subset \mathbb{R}$ and $p \in H^{1,q}_{loc}(G)$ such that $(p_n - c_n)$ converges in $H^{1,q}_{loc}(G)$ to p. The sequence (c_n) may be choosen independently of q.

In case q = 2, Theorem 1 follows immediately from Theorem 2. For $q \neq 2$, $1 < q < \infty$ we need

Theorem 3. a) There exists a constant $C = C(G,q,N) > 0$ such that for

$$p \in \hat{H}^{1,q}(G): \qquad C\|\nabla p\|_q \le \sup_{0 \neq \Phi \in \hat{H}^{1,q'}(G)} \frac{\langle \nabla p, \nabla \phi \rangle}{\|\nabla \Phi\|_{q'}} .$$

b) Given $F \in (\hat{H}^{1,q'}(G))^*$, then there exists one and only one $p \in \hat{H}^{1,q}(G)$ such that $F(\Phi) = \langle \nabla p, \nabla \Phi \rangle$ for $\Phi \in \hat{H}^{1,q'}(G)$ and $C\|\nabla p\|_q \le \|F\|_q \le \|\nabla p\|_q$.

The proof is quite elementary but tricky and will appear in [5], [6]. It follows the spirit of [4]. Clearly Theorem 3 gives the key for the solution of the weak Neumann problem in L^q : Let $\underline{f} \in L^q(G)^N$ such that $\langle \underline{f}, \nabla\varphi \rangle = 0$ for all $\varphi \in C_o^\infty(G)$. Then, by $F(\Phi) := \langle \underline{f}, \nabla \phi \rangle$ for $\phi \in \hat{H}^{1,q'}(G)$ an element of $(\hat{H}^{1,q'}(G))^*$ is defined and by Theorem 3 b) there exists $u \in \hat{H}^{1,q}(G)$ such that $\langle \nabla u, \nabla \phi \rangle = \langle \underline{f}, \nabla \phi \rangle$ holds for all $\phi \in \hat{H}^{1,q'}(G)$. If $\underline{f} \in C^1(\bar{G})$ and $\partial G \in C^2$ by elliptic regularity theory (compare e.g. [6]) we conclude $u \in C^2(G) \cap C^1(\bar{G})$ and $-\Delta u = -\text{div}\, \underline{f} = 0$, $\frac{\partial u}{\partial n}\Big|_{\partial G} = (\underline{f}, \underline{n})\Big|_{\partial G}$ where $\underline{n}(x)$ denotes the outward unit normal vector of ∂G. Therefore u is a weak L^q-solution of this Neumann problem.

From Theorems 2 and 3 Theorem 1 follows immediately. Observe that in Theorem 1, part c) we immediately see that the left hand side is contained in the right hand side. But the converse inclusion is by no means trivial. The naturalness of our method of proof of the Helmholtz decomposition via Theorem 3 follows from

Theorem 4: i) Assume that Theorem 2 and Theorem 3a) hold. Then Theorem 1, a)

- c) hold true.

ii) Assume that Theorem 1, a) - c) hold. Then Theorem 3, a) and b) hold true.

For other decomposition theorems related with the Stokes system too one has to establish an L^q-theory analogously to Theorem 3 for the Dirichletproblem for Δ. If G is bounded this was done in [4]. But for an exterior domain new phenomena occur [6].

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