ON CENTERS OF TYPE A AND B OF POLYNOMIAL SYSTEMS

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1 - We shall consider a planar system

$$(1.1) \qquad \dot{x} = X(x,y) , \qquad \dot{y} = Y(x,y) ,$$

where X, Y are real polynomials of $(x,y) \in \mathbb{R}^2$, relatively prime. The degree n of (1.1) is the maximum degree of X, Y.
Let S be a center of (1.1) and let N_S be the maximum neighborhood of S entirely covered by cycles surrounding S and no other singular point.
We say that S is a global center or a center of type A if $N_S = \mathbb{R}^2$.
For instance the origin 0 of $\mathbb{R}^2$ is a global center for

$$\dot{x} = y^n , \qquad \dot{y} = -x^n , \qquad n = 1,3,5,\ldots$$

A global center cannot exist for a quadratic system $(n = 2)$. This can be proved (cf. R. Conti [1]) by elementary geometric considerations based upon the wellknown fact that for a quadratic system the interior of any cycle is a convex set. Also, if (1.1) is a homogeneous system of even degree then there are no cycles.
These two facts suggested (cf. R. Conti [2]) the conjecture: "A polynomial system of even degree cannot have a global center".
Very recently M. Galeotti and M. Villarini ([3]) were able to prove the conjecture to be true. Actually, they proved more, namely:"A polynomial system of even degree has at least one unbounded trajectory".
To do so they made a detailed analysis of the vector field obtained from (1.1) by compactification on the Poincaré sphere.
The problem remains open of characterizing (by the coefficients of X, Y) systems of odd degree with a global center. A necessary condition is contained in [3].

2 - If $N_S \neq \mathbb{R}^2$ then it is easy to prove (cf. [2]) that the boundary ∂N_S of N_S is an invariant set, namely the finite union of singular points and open trajectories.

Then we can say that S is a center of type B if ∂N_S does not contain singular points, so that it is the finite union of open unbounded trajectories. Such centers actually exist for systems of any degree $n \geq 2$.

It is easy to show (cf. [2]) that for a given degree $n \geq 2$ the maximum number $k(n)$ of trajectories in ∂N_S cannot exceed $n+1$, i.e.,

(2.1) $\quad k(n) \leq n+1 \;, \quad n = 2,3,\ldots$

In a paper (cf. R. Conti [4]) to appear in a volume dedicated to Professor Otakar Borůvka, examples were given showing that

(2.2) $\quad n-1 \leq k(n) \;, \quad n = 2,3,\ldots$

also holds.

It can be easily proved (cf. [1]) that

(2.3) $\quad k(2) = 1 \;.$

(2.1), (2.2), (2.3) together suggest the

Conjecture 2.1: $\quad k(n) = n-1 \;, \quad n = 2,3,\ldots$

or, equivalently $k(n) \neq n, n+1, \quad n = 2,3,\ldots$

Finally, let $b(n)$ denote the maximum number of centers of type B for a system of degree n.

The examples of [4] show that

(2.4) $\quad n \leq b(n) \;, \quad n = 2,3,\ldots$

Since (cf. [1])

(2.5) $\quad b(2) = 2$

(2.4) and (2.5) suggest the

Conjecture 2.2: $b(n) = n$, $n = 2,3,\ldots$

References

[1] R. Conti, Centers of quadratic systems, Ricerche di Matematica, Suppl. Vol. XXXVI (1987), 117-126.

[2] R. Conti, Centers of polynomial systems, Ist. Mat. U.Dini 1987-88/17, Sept. 1988.

[3] M. Galeotti - M. Villarini, Some properties of planar polynomial systems of even degree, Ist. Mat. U.Dini, 1988-89/12, June 1989.

[4] R. Conti, On centers of type B of polynomial systems, to appear Archivum Mathematicum, Brno.