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Ferenc Fazekas

Remarks to chaotic vibration in nonlinear dynamical systems

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REMARKS TO CHAOTIC VIBRATION IN NONLINEAR DYNAMICAL SYSTEMS OF TECHNICS

FAZEKAS F., BUDAPEST, Hungary

0. Non-linear systems (nl.Ss) and their phenomena have basic importance in several sciences of our epoch. Fortunately, recent scientific results (e.g. ones about strange attractors) assured increasing efficiency for their investigations. It came to light, that nl.Ss - already at small degree of freedom ($f \geq 2$) and at certain constellation of parameters - show irregular, aperiodical, s.c. **chaotic behaviour** (ChB), more exactly stochastic motion at deterministic noise. This appears as very sensitive to the **initial conditions,** that is small changes of them cause large ones in trajectories of motion which are mixed finally in complicated manner. These are important also for the **engineers** at handling nl. vibrating (v.) Ss.

1. A very ample set of technical vibrating systems can be described by the non-linear differential equation (DE) [1-3]

$$m\ddot{x} + 2d_0\dot{x} + k_1x + k_3x^3 = f_0 + f_1 \sin \omega t + f_2 \cos \omega t \quad (0 < m, \omega) \qquad (1,1)$$

with axial swing out x (or angular one φ), time t, mass m (or inertial moment θ), reversion k_1 (lin.) and k_3 (cub.), viscose damping $d = 2d_0$ (lin.), amplitudes f_1, f_2, f_0 and circular frequency of exitation forces (period., const., resp.). - The eight **parameters** (m, $2d_0$, k_1, k_3; f_0, f_1, f_2, ω) can be reduced e.g. into five by special **transform** [11]

$$q = x\sqrt{|k_3|/m} \hat{=} x\sqrt{|\kappa_3|} ,\langle \kappa_1 = k_1/m , \quad \delta_0 = d_0/m , \qquad (1,2)$$

$$\alpha_i = -\frac{s_3}{m}\sqrt{|\kappa_3|}.f_i\rangle, \text{ namely (with } s_3 \hat{=} \text{sign } k_3)$$

$$s_3(\ddot{q} + 2\delta_0\dot{q} + \kappa_1 q_1) + q^3 = \alpha_0 + \alpha_1 \sin \omega t + \alpha_2 \cos \omega t . \qquad (1,3)$$

There are important for the practice

α) the Duffing's case (Dc) $k_1 > 0$, $k_3 > 0$; β_1) the general **pendulum's** case $k_1 > 0$, $k_3 < 0$ and β_2) the math. one $k_1 > 0$, $k_3 = -k_1/6$ (for small angles $|q| < \varepsilon$); γ) the Holmes' case (Hc) $k_1 < 0$, $k_3 > 0$) , etc. (1,4α-γ)

- Referring to our detailed investigation on this large system given by (1), there will be shortly treated only some special cases showing s.c.

chaotic **behaviours** and some **investigating methods** to them.

2. The non-linear **dissipative** ($\delta_0 > 0$, so E=H-D), free oscillator (at $\forall \alpha_i = 0$ and $s_3 = 1$) [1,6]

$$\ddot{q} + 2\delta_0\dot{q} + \kappa_1 q + q^3 = 0 \text{ , resp. } \hat{=} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} p \\ -2\delta_0 p - q(\kappa_1 + q^2) \end{bmatrix} \hat{=} f(q)$$

$$\langle p = \dot{q}, \quad \varrho^* \hat{=} [q,p] \rangle \qquad (2,1)$$

having the **derivatives** $F_i = F(\varrho_i) \hat{=} (\frac{df}{d\varrho^*}) = \begin{bmatrix} 0 & 1 \\ -\kappa_1 - 3q^2 & -2\delta_0 \end{bmatrix}_i$ (2,2)

at the fixed point(s) ϱ_i of rest $0 \hat{=} \dot{\varrho} = f(\varrho_i)$ and the **characteristical equations** $D_i(\lambda) \hat{=} |\lambda E - F_i| \hat{=} \lambda^2 + 2\delta_0 \lambda + (\kappa_1 + 3q^2) = 0$ (2,3)

- for linearizations in small at ϱ_i - results at $\kappa_1 \hat{=} \omega_0^2 > 0$ (Dc) by $_{(o)}\lambda = -\delta_0 \pm i\sqrt{\omega_0^2 - \delta_0^2}$ the sole $\varrho_0 = 0$ as an attractor (stable focus (for $\delta_0 < \omega_0$) node (for $\delta_0 > \omega_0$)); but at $\kappa_1 = -\omega^2 < 0$ (Hc) it ramifies, s.c. "bifurcates" by $_{(1,2)}\lambda = -\delta_0 \pm i\sqrt{2\omega_0^2 - \delta_0^2}$ into two attractors $\varrho_{1,2} = \pm\omega_0 e_1$ (stable focuses (for $\omega_0\sqrt{2} > \delta_0$) nodes (for $\omega_0\sqrt{2} < \delta_0$)) - with trajectories sundered by a separatrix (for $\delta_0 = \omega_0\sqrt{2}$) - and into a hyperbolic point (instable saddle)

$$\varrho_0 = 0 \quad [4,6] \qquad (2,4)$$

- In a **conservative** system ($\delta_0 = 0$, H=c), the sole stable centre ϱ_0 at $\kappa_1 \hat{=} \omega_0^2 > 0$ (Dc) by $_{(o)}\lambda = \pm i\omega_0$ "bifurcates" into two stable centres $\varrho_{1,2} = \pm\omega_0 e_1$ at $\kappa_1 \hat{=} -\omega_0^2 < 0$ (Hc) by $_{(1,2)}\lambda = \pm i\sqrt{2}\omega_0$ - with trajectories by the separatrix $p = \pm q\sqrt{\omega_0^2 - q^2/2}$ (at $H_0 = 0$) - and into an instable saddle

$$\varrho_0 = 0 \qquad (2,5)$$

3. A) The non-linear forced oscillator

$$\ddot{q} + 2\delta_0\dot{q} + \omega_0^2 q + \kappa_3 q^3 = \alpha_1 \sin \omega t + \alpha_2 \cos \omega t \hat{=}$$
$$\hat{=} \alpha \sin(\omega t + \varphi)$$
$$\langle q = x, \quad \kappa_1 = +\omega_0^2 > 0, \quad \kappa_3 = k_3/m \gtrless 0 \;;$$
$$\alpha_i = f_i/m, \quad \alpha_0 = 0 \rangle \qquad (3,1a,b)$$

responses to the first **approach** $q_1(t) = q_0 \sin \omega t$ ($0 < q_0$, $0 < \omega = ?$) - by Duffing's **suppositions** $_{(s)}q_{1o} \hat{=} q_0\omega^{-2}(\omega_0^2 + \frac{3}{4}\kappa_3 q_0^2 - \alpha_1/q_0) = q_0$, $_{(c)}q_{12} \hat{=} 2\delta_0\omega - \alpha_2/q_0 = 0$ - with the second **approach** and with circular frequency equations [2]

$$q_3(t) \hat{=} q_0 \sin \omega t - \kappa_3 q_0^3/36\omega^2 \sin 3\omega t \text{ with } \omega^2 = \qquad (3,2a,b)$$
$$= (\omega_0^2 + \tfrac{3}{4}3q_0^2) - \alpha_1/q_0 \text{ , } \quad 2\delta_0\omega = \alpha_2/q_0 < \operatorname{ctg}\varphi = \alpha_1/\alpha_2 =$$
$$= (\omega_0^2 - \omega^2 + \tfrac{3}{4}\kappa_3 q_0^2)/2\delta_0\omega \text{ , } \quad \alpha = \sqrt{\alpha_1^2 + \alpha_2^2} > \ldots . \qquad (3,3a-c)$$

The **resonance curve** $|q_0(\omega)|$ at $0 < 2\delta_0 < \varepsilon$ shows for the cases $\kappa_3 \gtrless 0$ a finite "beak" bending to a parabola and for $\kappa_3 = 0$ (lin.) the extremum $|q_0|_{max} = |\alpha_2|/2\delta_0\omega_0$. One can refer here to stable and instable solutions, to jumps, to hysteresis loop [11b].

B) For the **special case** $\omega_0 = 0$ (at $0 < 2\delta_0 = \delta$, $0 < \kappa_3 \ll 1$) (const.)

$$\ddot{q} + \delta \dot{q} + \kappa_3 q^3 = \alpha \sin(\omega t + \varphi) \qquad (3,4a)$$

a particular (approximate) solution of inhom. DE is - from (3,2) by derivation

$$\hat{z}_1(t) \mathrel{\hat{=}} \dot{q}_3(t) = q_0\omega \cos \omega t - \frac{\kappa_3 q_0^3}{12} \cos 3\omega t \quad \text{with}$$

$$\omega^2 = \frac{3}{4}\kappa_3 q_0^2 - \alpha_1/q_0 = \alpha_2^2/\delta^2 q_0 \quad ; \qquad (3,4b\text{-}c)$$

the general (such) one of hom. DE - got by Poisson's **small parametrical method**

$$(\ddot{q}_0 + \delta \dot{q}_0)_0 + \kappa_3(\ddot{q}_1 + \delta \dot{q}_1 + q_0)_0 + o(\kappa_3^3) \approx 0 \quad \text{at}$$

$$\dot{q}(t) \mathrel{\hat{=}} \dot{q}_0(t) + \kappa_3 \dot{q}_1(t) = z_{20} e^{-\delta t} - \kappa_3 \frac{z_{20}^3}{2\delta^4} e^{-3\delta t} \; . \qquad (3,5a\text{-}b)$$

The general (approximate) solution $z_2(t) \mathrel{\hat{=}} \dot{q}(t)$ of inhom. DE (3,4a) at an arbitrary initial value $\dot{q}_0(0) = z_{20}$ and its tendence at $t \to +\infty$ is as follows:

$$z_2(t) \mathrel{\hat{=}} \check{z}_2(t) + \hat{z}_2(t) \to 0 + \hat{z}_2(t) = \hat{z}_2(t+T) \qquad (3,5c)$$

so the periodic function $\hat{z}_2(t)$ is the s.c. **limit cycle** $\hat{G}_2$ of velocity $q(t)$ for asymptotic motion $z_2(t)$ of DE.

C) Making on the plan $F \ni z \mathrel{\hat{=}} (z_1, z_2) \doteq (q, \dot{q}) \mathrel{\hat{=}} \varrho$ a s.c. Poincaré-mapping [4] on the time series $t_i = t_0 + iT$ (at $0 \leqq t_0 < T = 2\pi/\omega$, $0 < \kappa_3 \ll 1$ and $0 < \varepsilon \mathrel{\hat{=}} e^{-\delta T} < 1$)

$$\varrho_1 \mathrel{\hat{=}} \varrho(t_1) \approx \hat{\varrho}(t_0) + \check{\varrho}(t_0)e^{-\delta T} \mathrel{\hat{=}} \hat{\varrho}_0 + \check{\varrho}_0 \varepsilon \; ,$$

$$\varrho_2 \mathrel{\hat{=}} \varrho(t_2) \approx \hat{\varrho}_0 + \check{\varrho}_0 \varepsilon^2, \ldots, \varrho_{n+1} \mathrel{\hat{=}} \varrho(t_{n+1}) \approx \hat{\varrho}_0 + \check{\varrho}_0 \varepsilon^{n+1} =$$

$$= \hat{\varrho}_0 + (\varrho_n - \hat{\varrho}_0) \mathrel{\hat{=}} P(\varrho_n) \to \hat{\varrho}_0 \mathrel{\hat{=}} P(\hat{\varrho}_0) \quad \text{at} \quad n \to \infty \; , \qquad (3,6)$$

consequently an arbitrary point $\hat{\varrho} \in \hat{G} \subset F$, so the whole limit cycle (LC) curve $\hat{\varrho}(t) : \hat{G} = P(\hat{G})$ too is a **stable** (fixed) **point/path** of the asymptotic motion on the plane $\varrho \mathrel{\hat{=}} (q, \dot{q})$ determined by the DE (3,4a). Remarkable that LC $\hat{\varrho}(t)$ $(0 \leqq t < T = 1)$ is a closed planar curve $\hat{G} \subset F$ with two loops (because of $\sin \omega t$ and $\sin 3\omega t$); but it is **projection** of the closed space curve $\hat{H}$ without loops in the phase space $S \ni \sigma \mathrel{\hat{=}} (q, \dot{q}, \omega t)$.

D) Decreasing the frictional parameter δ $(= \alpha_2/\omega q_0$, so increasing the circular frequency ω) under the values of certain **sequence** $\delta_1 > \delta_2 > \ldots > \delta_n > \ldots > \delta_\infty$ $(> \ldots)$, then sequential **point-bifurcations** $\hat{\varrho}_0 \to \langle \hat{\varrho}_1, \hat{\varrho}_2 \ldots$ (**period-duplications** of LCs)

$$F \supset \hat{G}_1 \mathrel{\hat{=}} \hat{G}_T \to \hat{G}_{2T} \mathrel{\hat{=}} \hat{G}_2 \to \ldots \to \hat{G}_n \to \ldots \to \hat{G}_\infty \quad (\to \text{chaos}) \; . \qquad (3,7)$$

Under δ_∞, the (general spatial) asymptotic motion $\sigma(t)$ becomes unperiodical, its trajectories $\sigma_{oi}^{(\infty)}(t)$ very sensitively depending from the initial values $\sigma_{oi}^{(\infty)}(t_o) = \sigma_{oi}^{(\infty)}$ adhere to a funny **strange attractor** (surface) $S \supset \hat{H}$: $\hat{\sigma}^{(\infty)}(t)$, similarly to the projected (planar) asympt. motion $\varrho^{(\infty)}(t)$, its trajectories $\varrho_i^{(\infty)}(t)$ adhering to the strange attractors (curve) $F \supset \hat{G}_\infty$: $\hat{\varrho}^{(\infty)}(t)$ (on which the points $\hat{\varrho}_i^{(\infty)}$ jump at random); practically, the forecasting of the asympt. motion $\varrho^{(n)}(t)$ and still more one of $\sigma^{(n)}(t)$ is impossible for $n > N$ [4].

4. A) In the former (mechanical) nl. vS (3.B-D), the developping of the chaos happened on the s.c. Feigenbaum-way (Fw.), namely through an infinite sequence r_n (with heaping value r_t) of **period-doubling bifurcations** (at the s.c. control parameter (cp.) $r \hat{=} \delta$). A similar chaotic formation presents itself in the (electronic) forced nl. vS Van der Pol given by the DE, or the equivalent SDEs

$$\ddot{u} - \mu(1-u^2)\dot{u} + u = a\,\mu\cos\,\omega t\ , \quad (\text{cp.: } r \hat{=} a) \qquad (4,1)$$

$$\dot{u} = \mu\left[v-(u^3/3-u)\right]\ , \quad \dot{v} = -u/\mu + a\cos\varphi\ , \quad \dot{\varphi} = \omega\ , \qquad (4,2a\text{-}c)$$

treated in detail in our [11c], together with variants. A such one describes the s.c. **heart-arithmy** meaning heartbeats with random variable time intervals ..... (4,3). - The most simple nl. SDEs of 3 variables defines the s.c. Rössler-**model** [4] :

$$\dot{x} = -(y+z), \quad \dot{y} = x+ay, \quad \dot{z} = b+xz-cz \quad (\text{cp.: } r \hat{=} c) \qquad (4,4a\text{-}c)$$

has also a chaotic advance on Fw., further much S too [7].

B) A possible other way guides - by increasing of r - at a certain r_t to the chaotic state; a typical example is the (hydrodynamical) Lorenz-**model** [4,7]. - A third s.c. Hopf-Landau-**way** has a sequence r_n without r_t , then a fourth s.c. Ruelle-Takens-Newhouse-**way** contents 2 (or 3) r-values before r_t (hydrodyn. ones), etc.

C) As in 3., one often studies the ChB by **Pm.** (of 1-,2-dim., with a cp. r), e.g. in 1-dim. case by $q_{t+1} = f(r,q_t)$ at q_0 and $t=0,1,2,\ldots$, or specially by the **logistic mapping** (Lm) [11e]

$$q_{t+1} = rq_t(1-q_t) \hat{=} f_L(r,q_t), \quad (0 \leq q_t \leq 1, \quad 1 < r < 4) \qquad (4,5)$$

and qualify its fixed points $\hat{q}_i \hat{=} f_L(r,\hat{q}_i) = 0,\ (r-1)/r$ with Ljapunov's **stability number** [11e] $\lambda_2 = \ln|f'_{Lq}(r,\hat{q}_2)| \leq 0$ at $1 < r \leq r_1$; similarly the $\hat{q}_2$'s bifurc.

$$\hat{q}_j \hat{=} f_L^{(2)}(r,\hat{q}_j) = \hat{q}_3,\ \hat{q}_4 \quad \text{with} \quad \lambda_j = 0{,}5\ \ln|f'^{2}_{Lq}(r,\hat{q}_j)| \leq 0$$
$$\text{at} \quad r_1 < r \leq r_2\ ;\ \ldots;\ r_\infty = 3{,}5699\ \ldots \qquad (4,6)$$

For r_k , one has the limit [11e]

$$\lim\,(r_k - r_{k-1})/(r_{k+1} - r_{k-1}) = 4{,}6690\ \ldots = \delta\ , \qquad (4,7a\text{-}b)$$

so $r_\infty - r_k = c\delta^{-k}$. This Feigenbaum's **constant** is universal, being for large set of 1- and higher dim. mappings [11d]. Remarks about **strange attractors** and ones of **fractal dim.** (Hausdorff) [11e].

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