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DIFFERENTIAL EQUATIONS, HIGHER MONOTONICITY AND PRINCIPAL PAIR OF SOLUTIONS

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1. Introduction.

There have been many investigations of the distribution of zeros of solutions of oscillatory equation

$$y'' + q(t)y = 0. \tag{q}$$

One approach to this problem is to compare the sequences $\{t_k\}$ of zeros of solutions y of (q) with the sequences $\{T_k\}$ of zeros of solutions Y of

$$Y'' + Q(t)Y = 0 \tag{Q}$$

where, in same sense the coefficients $Q(t)$ and $q(t)$ differ only slightly.

Special functions can be considered as solutions of suitable equations (q) or

$$y'' + a(t)y' + b(t)y = 0 \tag{a,b}$$

with very strong regularity conditions on coefficients. Namely the case when q, Q, a, b, A, B posses certain higher monotonicity properties is important. As an example we can consider the familiar Bessel equation

$$y'' + \frac{1}{t}y' + (1 - \frac{\nu^2}{t^2})y = 0 \qquad t > 0\,. \tag{1}$$

Here the function $a(t) = 1/t$ is completely monotonic on $(0,\infty)$ (i.e. $(-1)^j(1/t)^{(j)} \geq 0$ for $j = 0, 1, \ldots$ and we write $a(t) \in M_\infty(0,\infty)$) and $b(t)$ has completely monotonic derivative. Results describing the relation between some higher monotonicity properties of coefficients and corresponding ones of solutions are presented. Only the case of complete monotonicity is considered here.

2. Sequences $\{r_k^{(i)}\}_{k=0}^{\infty}$ and $\{R_k^{(j)}\}_{k=0}^{\infty}$.

LEMMA: ([4]) If $q' \in M_\infty(0,\infty)$, $0 < q(\infty) \leq Q(\infty) \leq \infty$ and $0 \leq (q - Q) \in M_\infty(0,\infty)$ then the condition

$$\int^\infty [\sqrt{q(t)} - \sqrt{Q(t)}]dt < \infty$$

is necessary and sufficient to ensure that, corresponding to the sequence $\{T_k\}_{k=0}^{\infty}$ of zeros of any solution of (Q) there is some solution of (q) whose sequence $\{t_k\}_{k=0}^{\infty}$ of zeros is such that

$$\{t_k - T_k\}_{k=0}^{\infty} \in M_{\infty,0}.$$

Remark that M_∞ denote the class of completely monotonic sequences i.e. sequences $\{t_k\}_{k=0}^{\infty}$ complying with $(-1)^j \Delta^j t_k \geq 0$ for $j, k = 0, 1, 2, \ldots$ Here $\Delta t_k := t_{k+1} - t_k$, $\Delta^0 t_k := t_k$. If in addition $\Delta t_k \to 0$ for $k \to \infty$, we write $\{t_k\} \in M_{\infty,0}$. For application see e.g. [2].

This result can be generalized in the following way. Consider the quantities

$$r_k^{(i)} := \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} w(t) \mid \exp[\tfrac{1}{2} \int a_i(t)dt] y^{(i)}(t) \mid^\lambda dt \; . \tag{2}$$

Here $\{t_k^{(i)}\}$ denotes the sequence of consecutive zeros of i-th derivatives of solution y or any other of (a,b), $\lambda > -1$, $w(t)$ is any completely monotonic function and

$$a_i(t) := a_i - b_i'/b_i, \qquad a \equiv a_0, b \equiv b_0 \tag{3}$$

$$b_i(t) := b_i + a_i' - ab_i'/b_i \qquad f_i(t) = b_i - \frac{1}{2}a_i' - \frac{1}{4}a_i^2 \; .$$

The special choice of i, λ, w in (2) gives $r_k^{(i)}$ a different geometrical (or other) meaning as e.g. Δt_k, $\Delta t_k^{(i)}$, $\Delta y^2(t_k')$, diameter of oscillating circle at the local extrema, $\int_{t_k}^{t_{k+1}} \mid y(t) \mid dt$, which enable us to describe more precisely the behaviour of $y(t)$.

Let $R_k^{(i)}$ denote the analogous quantity related to the equation (A,B). We can prove the following

THEOREM 1: Let $w(t)$, $W(t)$ be any completely monotonic functions satisfying $(W - w) \in M_\infty(0,\infty)$ and $W(\infty) = w(\infty)$. Let for some $i, j \in N$

$$f_i' \in M_\infty(0,\infty), \quad (f_i - F_j) \in M_\infty(0,\infty), \quad 0 < f_i(\infty) = F_j(\infty) \leq \infty \; .$$

Then the condition

$$\int^\infty [\sqrt{f_i(t)} - \sqrt{F_j(t)}]dt < \infty$$

is necessary and sufficient to ensure that corresponding to the sequence $\{R_k^{(i)}\}_{k=0}^{\infty}$ there is some solution $y(t)$ of (a,b) such that

$$\{R_k^{(j)} - r_k^{(i)}\}_{k=0}^{\infty} \in M_{\infty,0} \; . \tag{4}$$

For the proof see [5].

It can be proved, that the conditions of Theorem 1 are satisfied e.g. for Bessel equation in the case $i = j = 0$, $\frac{1}{2} < \nu_1 < \nu_2$ and (4) is valid for

$$y(t) = J_{\nu_1}(t), \qquad Y(t) = J_{\nu_2}(t).$$

For $\lambda = 1$ and $w(t) = 1/t$ $r_k^{(0)}$ denote the areas under the arches of the graph of a Bessel functions. On the suitable interval (α, ∞) we have

$$\{t_k - t'_k\} \in M_\infty \quad \{t'_k - t''_k\} \in M_\infty, \quad \{t_k - t''_k\} \in M_\infty.$$

Here t_k, t'_k, t''_k denote zeros of corresponding derivatives of the same or different Bessel function $C_\nu(t)$. Investigation in this direction was made by Borůvka , Lorch, Muldoon, Laforgia, Elbert, Došlá, Háčik and many others.

This theory is based on the transformation

$$y(t) = \sqrt{v(t)}u(s), \qquad dt/ds = v(t)$$

which convents (q) to the sine equation $y'' + y = 0$. Here $v(t) = y_1^2 + y_2^2$, (y_1, y_2) being the suitable pair of solutions of (q).

3. Principal Pair of Solutions.

More detailed investigation of the above mentioned transformation shows that the important role play certain special pairs (y_1, y_2) of solutions of (q).

In the case of nonoscillatoric equation (q) Hartman introduced some forty years ago the notion of principal solution - certain exceptional solution with some extremal properties. A natural question arise whether there exists some analogy for the oscillatoric case. Investigation in this direction was made recently together with A. Elbert and F. Neuman (see [1]).

The main idea is simple. One from the fundamental systems of solutions of $y'' + y = 0$ is the pair $(\cos t, \sin t)$ with the very nice property $\cos^2 t + \sin^2 t = 1$.In the case of Bessel equation is the function $v_\nu = t(J_\nu^2(t) + Y_\nu^2(t))$ for $| \nu | > \frac{1}{2}$ completely monotonic and for $| \nu | < \frac{1}{2}$ has similar monotonic properties. So $J_\nu(t)$, $Y_\nu(t)$ can be considered as a "good pair" while e.g. the fundamental system $J_\nu(t)$, $J_{-\nu}(t)$ cannot. We try to generalize this fact for a larger class of equations.

The direct calculation shows, that the function

$$v(t) = ay_1^2 + by_1y_2 + cy_2^2$$

y_1, y_2 being any solutions of (q) complies with the Mammana identity

$$M(v) := 2v''v - v'^2 + 4qv^2 = \text{const}.$$

and

$$M(ay_1^2 + by_1y_2 + cy_2^2) = (4ac - b^2)W^2, \quad W = y_1y_2' - y_1'y_2$$

$$M(v) = 0 \Leftrightarrow v = \bar{y}_1^2, \quad M(v) > 0 \Leftrightarrow v = \bar{y}_1^2 + \bar{y}_2^2, \quad M(v) < 0 \Leftrightarrow v = \bar{y}_1\bar{y}_2$$

for suitable $\bar{y}_1, \bar{y}_2$. We can prove the following

THEOREM 2: Let $M(v_0) = 1$ and let there exist $\lim_{t\to\infty} v_0(t) = l$ (> 0). Then $v_0(t)$) is unique and there is a pair (y_1, y_2) of solutions of oscillatoric equation (q) such that $v_0 = y_1^2(t) + y_2^2(t)$ and the pair (y_1, y_2) is unique up to its replacement by $(\alpha y_1 + \beta y_2, \gamma y_1 + \delta y_2)$ with orthogonal matrix $\left(\begin{smallmatrix}\alpha & b\\ \gamma & \delta\end{smallmatrix}\right)$.

DEFINITION: The above mentioned pair (y_1, y_2) we call *principal pair of solutions of* (q).

There are several conditions for $q(t)$ guaranteeing the existence of $\lim_{t\to\infty} v(t)$, e.g. monotonicity of $q(t)$.(For some other see e.[3])

But there is another way how to select certain exeptional pair (y_1, y_2) of solutions of (q). For any couple (y_1, y_2) with $W(y_1, y_2) = 1$ consider $v = ay_1^2 + by_1y_2 + cy_2^2$ with a, b, c such that $4ac - b^2 = 4$. This implies $\mathrm{M}(v) = 4$.

For such v consider the expression $\omega = ay_1'^2 + by_1'y_2' + cy_2'$ and suppose the existence of the limit

$$L(a,b,c) = \lim_{T\to\infty} \frac{\int_0^T (ay_1'^2 + by_1'y_2' + cy_2'^2)dt}{\int_0^T (y_1'^2 + y_2'^2)dt} \quad \text{for any} \quad a, b, c.$$

There is possible to prove the existence of a, b, c minimizing $L(a, b, c)$. This triplet defines the unique $v(t)$ and by means of this $v(t)$ we receive the *extremal pair* (y_1, y_2). Under certain conditions (e.g. the existence of $\lim_{t\to\infty} v^2(t)$), the extremal pair coincides with the principal pair.

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