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ON ANALYSIS OF QUASIHYPERBOLIC EQUATIONS OF LINEAR VISCOELASTICITY

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1. Introduction

When dealing with initial-boundary value problems of the linear viscoelasticity we arrive at the following class of equations

$$Au = \sum_{k=0}^{r} A_k D_t^k u = f^* , \quad x \in \Omega, \ t \in (0,\infty) , \tag{1}$$

where Ω is a bounded domain in R^2 with a smooth boundary S , A_k are strongly elliptic differential operators of order $2m$, m is one in the case of a viscoelastic membrane, 2 in the case of a viscoelastic plate and 4 in the case of a viscoelastic shell. Then u is a transverse displacement, a stress function or a shell function and $D_t^k = \partial^k/\partial t^k$.

In the case of membranes and plates, the right hand side of (1) assumes the form

$$f^* = \sum_{k=0}^{s} a_k D_t^k (f - \rho D_t^2 u) , \tag{2}$$

where a_k are material constants, $a_s = 1$, f is the transverse loading and ρ is the mass density per unit area element. For real materials $r = s$ or $r = s + 1$.

Operators A_k can assume also a tensor form

$$A_n(\,.\,) \equiv -C^n_{ijkl}(\,.\,)_{k,jl} , \quad i,j,k,l = 1,2,3 , \tag{3}$$

where we apply the summation rule with respect to double indices of space variables and a comma followed by indices denotes the differentiation with respect to space variables. C^n_{ijkl} are tensors of elastic and viscoelastic moduli and $\underline{u}$ is a vector of the displacement.

We shall consider Dirichlet boundary conditions and nonhomogeneous initial conditions. When $\rho = 0$ the equation (1) describes quasistatic problems and for $\rho \neq 0$ dynamic problems of the viscoelasticity. For $m = r + 1$ we arrive at equations

$$\rho D_t^{r+1} u + \sum_{k=0}^{r} B_k D_t^k u = \sum_{k=0}^{r-1} a_k D_t^k f = f^* , \tag{4}$$

where

$$B_k = A_k + a_{k-2}\rho .$$

For $r = s$ we have equations

$$\rho D_t^{r+2}u + a_{r-1}\rho D_t^{r+1}u + \sum_{k=0}^{r} B_k D_t^k u = \sum_{k=0}^{r-1} a_k D_t^k f = f^* . \tag{5}$$

Multiplying (1) by the inverse of the operator $\sum_{k=0}^{s} a_k D_t^k$ we arrrive for $s = r$ and homogeneous initial conditions with exception of u_0 and u_1 at

$$\rho D_t^2 u + A_r u + \int_0^t G_{r-1}(t-\tau)u(x,\tau)d\tau = f^*, \tag{6}$$

where $G_{r-1}(t)$ is an elliptic operator with respect to space variables. Similary for $r = s + 1$ we obtain

$$\rho D_t^2 u + A_r D_t u + (A_{r-1} - a_{r-2}A_r)u + \int_0^t G_{r-2}(t - \tau)u(x,\tau)d\tau = f^* , \tag{7}$$

where for real materials $A_{r-1} - a_{r-2}A_r$ and G_{r-2} are positive definite operators. Thus we have arrived at two different classes of equations.

2. *Variational formulation*

For variational formulation of problems under consideration we apply Laplace transform. Let $f(x,t) \in L_2(R^+, H^m, \sigma)$, what is the weigthed Hilbert space with the norm

$$\|f\|^2_{L_2(R^+,H^m,\sigma)} = \int_{-\infty}^{\infty} \|f\|^2_{H^m(\Omega)} e^{-2\sigma t}dt . \tag{8}$$

Then applying Laplace transform to (1) we arrive at

$$\tilde{B}\,\tilde{u} = \tilde{A}\,\tilde{u} + \rho\sum_{k=0}^{s} a_k p^{k+2}\,\tilde{u} = \sum_{k=0}^{r} p^k A_k\,\tilde{u} + \rho\sum_{k=0}^{s} a_k p^{k+2}\,\tilde{u} = \sum_{k=0}^{s} a_k p^k\,\tilde{f} + \tilde{f}_i = \tilde{F} , \tag{9}$$

where $\tilde{f}_i$ denotes Laplace transform of initial conditions. From the principle of nonnegative work we have proved [1], that in the case of real materials the operator A is for real nonnegative values of p positive definite and

$$\sum_{k=0}^{r} a_k p_k = 0 \tag{10}$$

cannot have positive roots. Then for real nonnegative values of p it holds

$$(B\,\tilde{u},\tilde{u}) \geq K\|\,\tilde{u}\,\|^2 , \quad K > 0 . \tag{11}$$

Thus we can construct the functional of the generalized potential energy

$$2\,\tilde{V}(\tilde{u}) = (B\,\tilde{u},\tilde{\bar{u}}) - 2(\tilde{f},\tilde{\bar{u}}) \qquad (12)$$

and formulate the following variational theorem

Theorem 1. The solution $\tilde{u}$ of (9) minimizes the functional of the generalized potential energy (12) for nonnegative real values of p and makes it stationary for other values of p. Conversely, the element which minimizes (12) for nonnegative real values of p gives the solution of (9). The solution of (1) is given by inverse Laplace transform.

Further we introduce complex Sobolev spaces of functions, which are parametrically dependent on the transform parameter p and analytic with respect to p in the right hand halfplane $p_\sigma^+ = \{p|\mathrm{Re}\; p \geq \sigma \geq 0\}$. We define the norm in $H^m(\Omega,\sigma)$ as

$$\|\,\tilde{u}\,\|^2_{H^m(\Omega,\sigma)} = \int_\Omega \sum_{|\alpha|\leq m} |D^\alpha\,\tilde{u}\,|^2 d\Omega\,, \qquad (13)$$

where α is a multiindex.

Now we associate with B a bilinear form and consider

$$H(\tilde{u},\tilde{v}) = \frac{1}{p^{s+2}}(B\,\tilde{u},\tilde{v}) = (\frac{\tilde{F}}{p^{s+2}},\tilde{v})\,. \qquad (14)$$

There exists such neighbourhood of real positive axis of p, where the bilinear form $H(\tilde{u},\tilde{v})$ fulfils the conditions of the Lax-Milgram theorem and we have :

Theorem 2. For $\tilde{F}\,/p^{s+2} \in H^{-m}(\Omega,\sigma) = (H_0^m(\Omega,\sigma))'$ there exists a unique weak solution $\tilde{u} \in H^m(\Omega,\sigma)$ and it holds

$$\|\,\tilde{u}\,\|_{H^m(\Omega,\sigma)} \leq C\|F/p^{s+2}\,\|_{H^{-m}(\Omega,\sigma)}\,. \qquad (15)$$

This estimate is valid for fixed values of $p \in p_\sigma^+$. When we want to derive global a priori estimates, we have to introduce new functional spaces.

3. Spaces with dominant mixed derivatives.

In contradistinction to elliptic equations the highest order of derivatives of partial differential equations for time dependent problems is different for different variables, e. g for time and space variables. Therefore for an analysis of time dependent problems it is convenient to introduce anisotropic Sobolev spaces. J. L. Lions and E. Magenes [2] have applied anisotropic Sobolev spaces with dominant ordinary derivatives for analysis of parabolic and hyperbolic equations. Howerer, in the case of hyperbolic equations they arrive at incompatibility of initial conditions and trace spaces.

We have dealt with analysis of quasiparabolic and quasihyperbolic equations [3] and have introduced spaces of analytic functions valued in Sobolev spaces for an analysis of Laplace transform of these equations and weighted anisotropic Sobolev spaces with dominant mixed derivatives for the analysis of original equations. As the considered equations have dominant mixed derivatives the introduced spaces are more convenient for their analysis then spaces with dominant ordinary derivatives.

We shall consider the spaces $K^{m,r}(\Omega,\sigma)$ of functions parametrically dependent on the transform parameter p and analytic in p_σ with the norm

$$\| \tilde{f} \|^2_{K^{m,r}(\Omega,\sigma)} = \sup_{p_1>\sigma} \int_{-\infty}^{\infty} (1+|p|^{2r}) \| \tilde{f}\,(p_1+ip_2)\|^2_{H^m(\Omega,\sigma)} dp_2 \,, \qquad (16)$$

where $p = p_1 + ip_2$. For $r = 0$ we arrive at Hardy spaces of functions valued in Sobolev spaces.

Simultaneously we introduce weighted anisotropic Sobolev spaces endowed with the norm

$$\|f\|^2_{H^{m,r}_{,0}(\Omega,R^+,\sigma)} = \int_0^\infty \left\{ \|f\|^2_{H^m} + \|D_t^r f\|^2_{H^m} \right\} e^{-2\sigma t} dt \,, \qquad (17)$$

where we assume that

$$D_t^k f(x,0) = 0 \,, \quad k = 0,1,\dots,r-1 \,. \qquad (18)$$

Then similarly as in [4] we can prove the following theorem .

Theorem 3. Laplace transform is an isomorphic mapping of $H^{m,r}_{,0}(\Omega,R^+,\sigma)$ onto $K^{m,r}(\Omega,\sigma)$.

When we deal only with the original time-space formulation of problems we can apply spaces $H^{m,r}(\Omega,R^+,\sigma)$ without the condition (18).

The norms (16) and (17) correspond to forms of operators for quasistatic problems of the linear viscoelasticity.

When we want to decide what spaces are convenient for special classes of differential equations, it is necessary to analyse properties of eigenvalues and eigenfunction expansions of solutions .

4. Analysis of hyperbolic equations with damping

We analyse the equation

$$(D_t^2 + kD_t + A_0)u = f^* \qquad (19)$$

where A_0 is a symmetric elliptic operator of the order $2m$. We consider homogeneous Dirichlet boundary conditions and following initial conditions

$$u(x,0) = u_0 \,, \quad D_t u(x,0) = u_1 \,. \qquad (20)$$

Then denoting by λ_n, X_n eigenvalues and orthonormal eigenfunctions of A_0 , respectively, Laplace transform of the solution assumes the form

$$\tilde{u} = \sum_{n=1}^{\infty}(p^2 + kp + \lambda_n)^{-1}[\tilde{f}_n + (p+k)u_{0n} + ku_{1n}]X_n \ , \qquad (21)$$

where $\tilde{f}_n = (\tilde{f}, X_n)$.

For nonlinear eigenvalues we have

$$p_{n1,2} = -\frac{1}{2}\left[k \pm (k^2 - 4\lambda_n)^{\frac{1}{2}}\right] \ . \qquad (22)$$

Thus, $p_{n1,2}$ are of the order m and for higher values of n assume complex conjugate values, which for $k = 0$ i.e. for hyperbolic equations are imaginary.

Hence by the inverse transform we arrive at

$$u = \sum_{n=1}^{\infty}\left\{\frac{1}{\omega_n}\int_0^t f_n(\tau)e^{-b(t-\tau)}\sin\omega_n(t-\tau)d\tau \right.$$
$$\left. + u_0 e^{-bt}\left(\frac{b}{\omega_n}\sin\omega_n t + \cos\omega_n t\right) + \frac{u_1}{\omega_n}e^{-bt}\sin\omega_n t\right\} X_n \ , \qquad (23)$$

where we have denoted

$$b = \frac{1}{2}k \ , \ \omega_n = (\lambda_n^2 - k^2/4)^{\frac{1}{2}} \qquad (24)$$

Now, as also hyperbolic equations can be written in the form with mixed derivatives, we introduce the following norm

$$\|f\|^2_{H^{2m,2}} = \int_0^{\infty}\int_{\Omega}\left(|f| + |D^{2m}D_t^2 f|\right)^2 e^{-2\sigma t}d\Omega dt \qquad (25)$$

We denote by χ_n^{2m} and φ_n eigenvalues and orthonormal eigenfunctions of the operator $D^{2m}(\ .\ )$, respectively. When, applying the weighted Fourier transform

$$\tilde{f} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(t)e^{-(\sigma+is)t}dt \qquad (26)$$

the norm can be written in the form

$$\|\tilde{f}\|^2_{H^{2m,2}} = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}[1 + \chi_n^{2m}(\sigma^2 + s^2)]^2|\tilde{f}_n|^2 ds. \qquad (27)$$

For analysis of the considered equation we restrict $H^{2m,2}$ to the subspace, for which $s \Leftrightarrow \chi^m$, i.e., for which a derivative with respect to the time variable corresponds to m-derivates with respect to space variables.

Now applying the usual procedure we can derive the following inequality for $t = 0$.

$$\|D_t^j f\|^2_{H^{\mu}(\Omega)} \le c^2\int_0^{\infty}\int_{\Omega}\left(|f| + |D^{2mk}D_t^{2k}f|\right)^2 e^{-2\sigma t}dtd\Omega$$
$$\le c\|f\|^2_{H^{2mk,2k}} \ , \qquad (28)$$

where

$$\mu = 2mk - j \qquad (29)$$

Analysis of trace properties for $x \in S$ are similar to that for anisotropic Sobolev spaces with dominant ordinary derivatives [2] . Then we have

$$\|D_\nu^j f\|_{H^{\mu_j,\varrho_j}(S,R^+,\sigma)} \leq c\|f\|_{H^{2mk,2k}(\Omega,R^+,\sigma)} , \tag{30}$$

where

$$\mu_j = 2mk - j - 1/2 , \quad \varrho_j = 2k - (j - 1/2)/m \tag{31}$$

for $j < 2mk - 1/2$. ν is an exterior normal to S .

Hence we can formulate

Theorem 4. Let $j \leq 2mk$, then $f \to D_t^j f$ are continuous linear mappings of

$$H^{2mk,2k}(\Omega, r^+, \sigma) \to H^{2mk-j}(\Omega) \tag{32}$$

and let $j < 2mk - 1/2$, then $f \to D_\nu^j f$ are continuous mappings of

$$H^{2mk,2k}(\Omega, R^+, \sigma) \to H^{2mk-j-1/2,2k-(j-1/2)/m}(S, R^+, \sigma) . \tag{33}$$

The functions $D_\nu^j f$ and $D_t^l f(x, 0)$ satisfy additional relations, called compatibility relations

$$D_t^l(D_\nu^j f|_S)_{t=0} = D_\nu^j(D_t^l f(x, 0))|_S . \tag{34}$$

In fact according to (33) it holds

$$D_\nu^j f|_S \in H^{2mk-j-1/2,2k-(j-1/2)/m}(S, R^+, \sigma) \tag{35}$$

and then applying (32) we have :

$$D_t^l(D_\nu^j f|_S)|_{t=0} \in H^{2mk-j-lm-1/2}(S) , \tag{36}$$

where it should hold $2mk - j - lm - 1/2 > 0$.

Hence

$$f \to D_t^l D_\nu^j f|_{S,t=0} \tag{37}$$

is a continuous mapping of

$$H^{2mk,2k}(\Omega, R^+, \sigma) \to H^{2mk-j-lm-1/2}(S) . \tag{38}$$

Similarly, according to (32)

$$D_t^l f|_{t=0} \to H^{2mk-lm}(\Omega) \tag{39}$$

therefore

$$D_\nu^j(D_j^l f|_{t=0})|_S \in H^{2mk-lm-j-1/2}(S) . \tag{40}$$

Then we can prove

Theorem 5. Let $f \in H^{0,0}(\Omega, R^+, \sigma)$, $u_0 \in H^{2m}(\Omega), u_1 \in H^m(\Omega)$ and boundary values

$$g_j \in H^{2m-j-1/2,2-(j-1/2)/m}(S, R^+, \sigma) , \tag{41}$$

$0 \leq j \leq m-1$, with the compatibility relations such that there exists a $w \in H^{2m,2}(\Omega, R^+, \sigma)$ satisfying

$$D_\nu^j w|_S = g_j , \quad 0 \leq j \leq m-1 \tag{42}$$

and

$$w(x,0) = u_0(x) , \quad D_t w(x,0) = u_1(x) . \tag{43}$$

Then the solution of problem (19), (20) with Dirichlet boundary conditions (41) satisfies

$$u \in H^{2m,2}(\Omega, R^+, \sigma) . \tag{44}$$

When we apply anisotropic Sobolev spaces with dominant ordinary derivatives $H^{2m,2}(\Omega; 0, T)$ used by J. L. Lions and E. Magenes (2) for analysis of hyperbolic equations, we have to consider the following relations for trace spaces

$$D_t^l f(x,0) \in H^{(2-l-1/2)m}(\Omega) \tag{45}$$

and

$$D_\nu^j f(s,t) \in H^{2m-j-1/2,2-(j-1/2)/m}(S; 0, T) . \tag{46}$$

Then we have to consider initial conditions

$$u_0(x) \in H^{3m/2}(\Omega) , \; u_1(x) \in H^{m/2}(\Omega). \tag{47}$$

However also in the case of hyperbolic equations when applying weighted anisotropic spaces with dominant mixed derivatives we arrive at compatibility between initial conditions and trace spaces.

5. Analysis of quasihyperbolic equations

When dealing with this analysis we restrict ourselves to the class of equations with $r = s$.

We consider the equation

$$(D_t^3 + kD_t^2 + A_1 D_t + A_0)u = f^* , \tag{48}$$

where A_1 and A_0 are symmetric elliptic operators of $2m$ order. We consider homogeneous Dirichlet boundary conditions and the following initial conditions

$$u(x,0) = u_0, \; D_t u(x,0) = u_1, \; D_t^2 u(x,0) = u_2 . \tag{49}$$

Then the Laplace transform yields

$$(p^3 + kp^2 + pA_1 + A_0)\,\tilde{u}$$
$$= \tilde{f}^* + (p^2 + pk + A_1)u_0 + (p + k)u_1 + u_2 \,. \tag{50}$$

For the sake of simplicity we assume that A_1 and A_0 are spectrally similar. We denote their eigenvalues by λ_{1n} and λ_{0n} and assume that eigenvectors X_n are orthonormal. Then eigenvector expansion of the solution can be written in the form

$$\tilde{u} = \sum_{n=1}^{\infty} \frac{1}{p^3 + kp^2 + \lambda_{1n}p + \lambda_{0n}} [\tilde{f}_n^* + (p^2 + pk + \lambda_{1n})u_{0,n}$$
$$+ (p+k)u_{1,n} + u_{2,n}]X_n \,. \tag{51}$$

For an analysis of (51) it is necessary to find nonlinear eigenvalues, which are the roots of the equations

$$p^3 + kp^2 + \lambda_{1n}p + \lambda_{0n} = 0 \,. \tag{52}$$

It is possible to prove that the real roots of these equations p_{1n} have a finite limit

$$\lim_{n\to\infty} p_{1n} = c_n(\lambda_n) < \infty \tag{53}$$

and thus are of order 0 . The remaining two roots are complex conjugate. Their real parts, have a finite limit and their imaginary parts have an infinite limit of order m as n tends to infinity.

Then the norm of the space $H^{2m,3}$ convenient for analysis of this class of equations can be written in the form

$$||f||^2_{H^{2m,3}(\Omega,R+,\sigma)}$$
$$= \sum_{\alpha=2m} \int_0^{\infty} \int_{\Omega} \left\{|f|^2 + |D_t D^{\alpha} f|^2 + |D_t^3 f|^2\right\} e^{-2\sigma t} d\Omega dt, \tag{54}$$

where α is a multindex.

For analysis of trace properties of thase spaces at $t = 0$, we apply an equivalent norm

$$||f||^2_{H^{2m,3}} = \sum_{\alpha=2m} \int_0^{\infty} \int_{\Omega} (|f| + |D_t D^{\alpha} f| + |D_t^3 f|)^2 e^{-2\sigma t} d\Omega dt \,. \tag{55}$$

When denoting by λ_n^{2m} and ϕ_n eigenvalues and orthonormal eigenfunctions of the operator D^{α} and applying the weighted Fourier transform (24) the norm (53) can be written in the form

$$||\, \tilde{f} \,||^2_{H^{2m,3}} = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} (1 + |-\sigma + is|\lambda_n^{2m} + |-\sigma + is|^3)^2 |\, \tilde{f}_n \,|^2 ds \,. \tag{56}$$

The roots of the equation

$$1 + (-\sigma + is)\lambda_n^{2m} + (-\sigma + is)^3 = 0 \qquad (57)$$

have the same limit properties as the roots of (50) and can be written in the form

$$(-\sigma + is)_1 = c_{0n}(\lambda_n^{2m})$$
$$(-\sigma + is)_{2,3} = -\frac{1}{2}c_{0n}(\lambda_n^{2m}) \pm ic_{mn}(\lambda_n^{2m}) \ , \qquad (58)$$

where c_{0n} is of order 0 and c_{mn} is of order m .

Then the norm assumes the form

$$\| \tilde{f} \|^2_{H^{2m,3}} = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} [b_{0n} + (s + c_{mn})^2][b_{0n}$$
$$+ (s - c_{mn})^2](s^2 + b_{1n}^2) | \tilde{f}_n |^2 ds, \qquad (59)$$

where $b_{0n} = -\sigma + 1/2c_{0n}$, $b_{1n} = \sigma + c_{0n}$.

Now using the common approach we can prove the following inequality

$$\|D_t^j f\|_{H^\mu(\Omega)} \leq c^2 \|f\|_{H^{2m,3}(\Omega, R^+, \sigma)} \ , \qquad (60)$$

where

$$\mu = 2m - j \ . \qquad (61)$$

Analysis of the solution (52) and of the proposed norm (57) shows that the solution is composed of functions belonging to $H^{2mk,2k}$ and to $H^{2m,\infty}$, to which terms with $1/(p - c_n)$ belong.

For traces of $H^{2m,\infty}$ it holds

$$D_t^j f(x, 0) \in H^{2m}(\Omega) \qquad (62)$$

and

$$D_\nu^j f|_S \in H^{2m-j-1/2,\infty} \ . \qquad (63)$$

Thus for analysis of the equation (46) it is convenient to apply spaces $H^{3m,3}(\Omega, R^+, \sigma)$.

Then we can prove

Theorem 6. Let $f^* \in H^{0,0}(\Omega, R^+, \sigma)$, $u_0 \in H^{3m}(\Omega)$, $u_1 \in H^{2m}(\Omega)$, $u_2 \in H^m(\Omega)$ and $g_j \in H^{3m-j-1/2}(S, R^+, \sigma)$, $0 \leq j \leq m-1$. Then there exists a solution of (46) $u \in H^{3m,3}(\Omega, R^+, \sigma)$.

Conclusion

We have shown that weighted anisotropic functional spaces $H^{2mj,2j}(\Omega, r^+, \sigma)$ with dominant mixed derivatives are convenient for analysis of quvasihyperbolic equations. They can be applied also for analysis of hyperbolic equation . The analysis in these spaces leads to a compatibility between initial conditions and trace spaces for $t = 0$.

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