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THE DIRICHLET PROBLEM
FOR SUBLAPLACIANS ON NILPOTENT LIE GROUPS - GEOMETRIC CRITERIA FOR REGULARITY

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In 1969 J. M. Bony [1] studied partial differential operators of second order

\[ L = X_1^2 + \cdots + X_n^2 \]

on \( \mathbb{R}^m \) where \( X_1, \ldots, X_n \) are \( C^\infty \)-vector fields. Let us note three simple examples:

1. \( n = m, X = \frac{\partial}{\partial x_1} : \quad L = \Delta \) (Laplace operator)

2. \( n = m = 2, X_1 = \frac{\partial}{\partial x}, X_2 = x \frac{\partial}{\partial y} : \quad L = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} \) (Grushin operator)

3. \( n = 2, m = 3, X_1 = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial z}, X_2 = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z} : \quad L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} + 4y \frac{\partial^2}{\partial x \partial z} - 4x \frac{\partial^2}{\partial y \partial z} + 4(x^2 + y^2) \frac{\partial}{\partial z} = \Delta_K \) (Laplace-Kohn operator)

Bony has shown the following: If the rank of the Lie algebra \( \mathcal{L}(X_1, \ldots, X_n) \) is equal to \( m \) at each \( x \in \mathbb{R}^m \) then \( \mathcal{H}_L(U) := \{ h \in C^\infty(U) : Lh = 0 \}, U \text{ open } \subset \mathbb{R}^m \), yields a Brelot space \( (X, \mathcal{H}_L) \).

Clearly Bony’s hypothesis is satisfied in each of the preceding simple examples \([X_1, X_2] = \frac{\partial}{\partial y} \) in (2), \([X_1, X_2] = -4 \frac{\partial}{\partial x} \) in (3)). In proving his result Bony had to find a base of regular sets. We recall that a relatively compact open set \( U \) is regular (with respect to \( L \) if for every \( f \in C^\infty(\partial U) \) there exists a unique function \( h \in C^\infty(\overline{U}) \) such that \( h|_{\partial U} = f \) and \( h|_U \in \mathcal{H}_L(U) \). Bony has shown that certain very flat lenticular sets are regular. But how about general criteria for regularity? Of course, having the harmonic space \( (\mathbb{R}^m, \mathcal{H}_L) \) it is well known that an open set \( U \subset \mathbb{R}^m \) (for which \( \overline{U} \) is contained in a \( \mathcal{P} \)-set) is regular if and only if the complement of \( U \) is not thin at any \( z \in \partial U \).

Now in many cases geometric properties of a set permit a decision on the thinness of the set at a point. This leads to geometric criteria for the regularity of open sets.

Given \( 0 < \alpha < \infty \), we shall say that \( U \) satisfies a pointwise exterior \( \alpha \)-Hölder condition if for every \( z \in \partial U \) there exists an isometry \( T \) of \( \mathbb{R}^m \) and \( \varrho > 0 \) such that \( T(z) = 0 \) and the \( \alpha \)-Hölder cone

\( \{ x \in \mathbb{R}^m : 0 < x_m < \varrho, (x_1^2 + \cdots + x_{m-1}^2)^{\alpha/2} < \varrho x_m \} \)

is contained in the complement of \( T(U) \). Looking at certain model cases we shall see that an exterior \((r - \varepsilon)\)-Hölder condition is in general not sufficient for regularity of \( U \).

We fix a real finite dimensional Lie algebra \( \mathcal{N} = V^1 \oplus \cdots \oplus V^r \) such that \([V^j, V^k] = V^{j+k} \neq \{0\} \) if \( k + j \leq r \), and \([V^j, V^k] = 0 \) if \( k + j > r \). Then \( \mathcal{N} \) is nilpotent of step \( r \) and \( V^1 \) generates \( \mathcal{N} \).

Example: Given a natural \( k > 2 \), let \( \mathcal{N} \) be the set of all upper triangular \((k \times k)\)-matrices \( (a_{pq}) \) \( (a_{pq} = 0 \) for all \( p > q \) \) and define a product on \( \mathcal{N} \) by \([A, B] = AB - BA \) (where \( AB \) denotes the usual matrix product of \( A \) and \( B \)). Then \( \mathcal{N} = V^1 \oplus \cdots \oplus V^{k-1} \) where \( V^i = \{(a_{pq}) \in \mathcal{N} : a_{pq} = 0 \) if \( p + i \neq q \} \), \( \dim V^i = k - i \), \( \dim \mathcal{N} = k \frac{(k-1)}{2} \).

In the general case let \( n_i := \dim V^i, 1 \leq i \leq r \). Then \( n_i := \dim \mathcal{N} = n_1 + \cdots + n_r \). Fixing a basis \( \{Y_{ij} : 1 \leq j \leq n_i\} \) for each \( V^i \) we identify \( (x_{ij}) \in \mathbb{R}^m \) with \( \Sigma x_{ij}Y_{ij} \in \mathcal{N} \). Using the map \( \exp \) from \( \mathcal{N} \) to the corresponding simply connected Lie group we obtain a product \( x \cdot y \in \mathbb{R}^m \) for \( x, y \in \mathbb{R}^m \) by \( \exp(x \cdot y) = \exp(x) \exp(y) \). Then \( (\mathbb{R}^m, \cdot) \) is a group such that \( x \cdot (-x) = 0 \) for every \( x \in \mathbb{R}^m \). By the Campbell-Hausdorff formula

\[ (x \cdot y)_{ij} = x_{ij} + y_{ij} + p_{ij}(x, y) \]
such that \( p_{ij}(x,y) \) is a linear combination of monomials \( z_{k_1}l_1 \cdots z_{k_n}l_n \) where each \( z \) is \( x \) or \( y \) and \( \sum_{j=1}^n k_j = i \). In particular, the Lebesgue measure \( \lambda^m \) on \( \mathbb{R}^m \) is invariant under left translations \( l_u : x \mapsto u \cdot x \), \( u \in \mathbb{R}^m \).

We now define left invariant vector fields \( X_1, \ldots, X_n \) on \( \mathbb{R}^m \) by
\[
X_j f(0) = \frac{\partial f}{\partial z_j}(0), \quad X_j f(u) = X_j f(l_u(0)).
\]
Then
\[
L = X_1^2 + \cdots + X_n^2
\]
is the corresponding sublaplacian on \( \mathbb{R}^m \). It is the unique left invariant differential operator satisfying
\[
L f(0) = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(0).
\]
Since \( \mathcal{N} \) is generated by \( Y_{11}, \ldots, Y_{1n_1} \), we have \( \mathcal{L}(X_1, \ldots, X_n)(0) = \mathcal{N} \) and hence by left translation \( \mathcal{L}(X_1, \ldots, X_n)(x) = \mathcal{N} \) for every \( x \in \mathbb{R}^m \). So \( L \) satisfies Bony’s hypothesis.

Note that \( n_1 \) may be much smaller than \( m \): In our example of triangular matrices, \( \frac{n_1}{m} = \frac{2}{k} \). In the case \( k = 3 \) we have \( m = 3 \),
\[
(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)),
\]
and hence
\[
X_1 = \frac{\partial}{\partial x} - \frac{1}{2}y \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{1}{2}x \frac{\partial}{\partial z}.
\]
Replacing \( z \) by \(-4z\) we obtain the Heisenberg group \( \mathcal{H}_1 \) and \( \Delta_K \).

We have natural dilations \( \delta_R \), \( R > 0 \):
\[
\delta_R((x_{ij})) = (R^i x_{ij}).
\]
Clearly, \( \delta_R(x+y) = \delta_R(x) + \delta_R(y) \) and by the Campbell–Hausdorff formula \( \delta_R(x \cdot y) = \delta_R(x) \cdot \delta_R(y) \).

As a consequence \( L \) is homogeneous of order 2, i.e.,
\[
L(f \circ \delta_R) = R^2(L f) \circ \delta_R.
\]
In the following let us assume that the homogeneous dimension
\[
Q = n_1 + 2n_2 + \cdots + rn_r
\]
is at least 3. Then there is a symmetric Green function satisfying
\[
G(\delta_R(x), 0) = R^{2-Q} G(x, 0).
\]
Defining
\[
\|x\| := \max_{i,j} |x_{ij}|^{1/i} \quad (x \in \mathbb{R}^m)
\]
we have \( \|\delta_R(x)\| = R \|x\| \). Therefore there exists a constant \( C > 0 \) such that
\[
C^{-1} \|x\|^{2-Q} \leq G(x, 0) \leq C \|x\|^{2-Q}.
\]
Indeed, having such inequalities on the compact set \( \{ x \in \mathbb{R}^m : \|x\| = 1 \} \) we obtain these inequalities on \( \mathbb{R}^m \) since \( G(\delta_R(x), 0) = R^{2-Q} G(x, 0) \) and \( \|\delta_R(x)\| = R \|x\| \). Since \( L \) is invariant under left translations, \( G(ux, uy) = G(x, y) \) for all \( u, x, y \in \mathbb{R}^m \). Defining
\[
d(x, y) := \|x^{-1} \cdot y\| \quad (x, y \in \mathbb{R}^m)
\]
we conclude that
\[
C^{-1} d^{2-Q} \leq G \leq C d^{2-Q}.
\]
The Campbell–Hausdorff formula yields that \( d \) is almost a distance on \( \mathbb{R}^m \). We have \( d(x, y) > 0 \) if \( x \neq y \), \( d(x, y) = d(y, x) \) and \( d(x, y) \leq D(d(x, z) + d(z, y)) \) for all \( x, y, z \in \mathbb{R}^m \) with a constant \( D = D(\mathcal{N}) \). H. Hueber [3] has shown that such a property is sufficient to obtain a Wiener criterion for regularity:
Theorem. Let $E$ be a Borel subset of $\mathbb{R}^m$, $0 < \eta < 1$, and define

$$E(\eta) = \{x \in E : \|x\| < \eta\}, \quad E_s = E(\eta) \setminus E(\eta + 1).$$

Then the following statements are equivalent:

1. $E$ is thin at $0$.
2. $\sum_{n \in \mathbb{N}} R_1^{E_\eta}(0) < \infty$.
3. $\sum_{n \in \mathbb{N}} \eta^{(2-Q)s} \text{cap}(E_s) < \infty$.
4. $\sum_{n \in \mathbb{N}} \eta^{(2-Q)s} \text{cap}(E(\eta)) < \infty$.

Of course, the capacity of a Borel set $A \subset \mathbb{R}^m$ is given by

$$\text{cap}(A) = \sup\{\mu(\mathbb{R}^m) : G\mu \leq 1, \mu(CA) = 0\},$$

and we have

$$\text{cap}(y \cdot A) = \text{cap}(A), \quad \text{cap}(\delta R(A)) = R^{q-2} \text{cap}(A).$$

Corollary. If $0 < \eta < 1$ and $\delta_\eta(E) \subset E$ then $E$ is thin at $0$ if and only if $\text{cap}(E) = 0$.

Let

$$I = \{(x_{ij}) \in \mathbb{R}^m : |x_{ij}| \leq 1 \text{ for all } i, j\}.$$

For every $0 < \gamma \leq 1$, let

$$V_\gamma = \{x \in I : x_{11} = 0, |x_{12}| \leq \gamma\}.$$

Using $(m-1)$-dimensional Lebesgue measure on $V_\gamma$ it can be shown that

$$\text{cap}V_\gamma \approx \frac{1}{1-\ln \gamma}$$

(where $a_\gamma \approx b_\gamma$ means that there is $C > 0$ such that $C^{-1} b_\gamma \leq a_\gamma \leq C b_\gamma$ for all $\gamma$). For every $0 < \gamma \leq 1$, $1 \leq k \leq r$ and $1 \leq l \leq n_k$ let

$$W_{\gamma}^{kl} = \{x \in I : |x_{ij}| \leq \gamma \text{ for all } 1 \leq i \leq k, 1 \leq j \leq n_k, j \leq l \text{ if } i = k\}.$$

Using Lebesgue measure $\lambda^m$ it can be shown that if $k \geq 2$ or $l \geq 3$ then

$$\text{cap}W_{\gamma}^{kl} \approx \begin{cases} \frac{\lambda^m(W_{\gamma}^{kl})}{\gamma^l}, & n_1 \geq 3, \\ \frac{\lambda^m(W_{\gamma}^{kl})}{\gamma^n(1-\ln \gamma)}, & n_1 = 2. \end{cases}$$

These estimates allow us to prove the following geometric criteria for regularity:

**Proposition.** Suppose that $Q \geq 4$, $n_1 \geq 3$, $0 < \varepsilon < 1$ and $\beta > 0$. Then the set

$$E = \{x \in \mathbb{R}^m : \beta|x|^{1-\varepsilon} \leq x_{rn_r} \text{ for } j = 1, 2, 3\}$$

is thin at $0$.

**Proposition.** For all $\beta, p > 0$ the set

$$E = \{x \in \mathbb{R}^m : \beta|x_{ij}|^{1} \leq x_{rn_r} \leq 1/\beta \text{ for all } (i,j) \neq (r,n_r), x_{11} = 0, \beta|x_{12}|^{p} \leq x_{rn_r}\}$$

is not thin at $0$. 
Fix $\alpha, \beta > 0$ and let

$$A = \{ x \in \mathbb{R}^m : \left( \sum_{(i,j) \neq (r,n_r)} x_{ij}^2 \right)^{\alpha/2} \leq \beta x_{nn_r} \}.$$  

**Theorem.** Suppose that $n_1 \geq 3$, $Q \geq 4$. Then $A$ is thin at 0 if and only if $\alpha < r$.

**Theorem.** Suppose that $n_1 = 2$ and $r \geq 3$. Then $A$ is thin at 0 if and only if $\alpha < \frac{r}{2}$.

**Theorem.** An outer ball condition is sufficient for regularity if and only if $r \leq 2$ or $n_1 = 2$ and $r \leq 4$.

Details can be found in [2].

**REFERENCES**

