Oldřich John; Jana Stará

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# ON THE REGULARITY AND NON-REGULARITY OF ELLIPTIC AND PARABOLIC SYSTEMS

JOHN O., STARÁ J., PRAGUE, Czechoslovakia

In our lecture we want to answer some questions connected with the properties of weak solutions of elliptic and parabolic systems, both the linear and the quasilinear ones. We are going to present our results in the context of some known facts about the regularity and non-regularity.

1. ELLIPTIC SYSTEMS. In what follows we denote as m the number of equations, n- the number of variables. So $u(x) = [u^1(x_1,\dots,x_n),\dots, u^m(x_1,\dots,x_n)]$. The Latin indices i,j run from 1 to m, Greek ones $\alpha$, $\beta$ from 1 to n. Denote further $D_\alpha = \partial/\partial x_\alpha$. Throughout the paper, the coefficient matrices $A = \{A_{ij}^{\alpha\beta}\}$ are supposed to be symmetric, i.e., $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. In the whole text, Einstein summation convention is used.

The linear elliptic system with bounded and measurable coefficients on the open subset $\Omega \subset R^n$ is of the form

$$(1) \quad D_\alpha(A_{ij}^{\alpha\beta}(x) D_\beta u^j) = 0, \; i=1,\dots,m,$$

$$(2) \quad A_{ij}^{\alpha\beta} \in L_\infty(\Omega),$$

$$(3) \quad A_{ij}^{\alpha\beta}(x)\, \xi_\alpha^i\, \xi_\beta^j \geqq \lambda |\xi|^2, \; (\lambda > 0).$$

A function u is said to be a weak solution of the system if (I) $u \in W_{2,loc}^1(\Omega)$, (II) $u \in L_\infty(\Omega)$, (III) u satisfies (1) in a sense of distributions.

In case that each weak solution of the system is locally $\mu$- Hölder continuous on $\Omega$ with some $0 < \mu \leqq 1$ we say that the system is regular. So if there exists any weak solution of the system which is not locally Hölder continuous on $\Omega$, then we speak about non-regularity.

Enlist here some important regularity results for the system (1) - (3). According to the classical results of C.B.Morrey [1], A.Douglis, L.Nirenberg [2] every system is regular provided $A_{ij}^{\alpha\beta}$ are continuous on $\Omega$. Moreover, the continuity of coefficients at the point $x^0$ implies the Hölder continuity of the weak solution u in some neighbourhood of $x^0$. In case of one equation (m=1) the regularity always holds It is a famous result of E.De Giorgi [3] and J.Nash [4]. Further, the system is regular in case of two variables (n=2).

The regularity result of another type can be formulated by means of algebraic condition. Let

$$(4) \qquad \lambda_0 |\xi|^2 \leqq A_{ij}^{\alpha\beta}(x)\, \xi_\alpha^i \xi_\beta^j \leqq \lambda_1 |\xi|^2, \quad (\lambda_0 > 0)$$

for all $\xi \in R^{nm}$ and almost all $x \in \Omega$. Define

$$(5) \qquad K(n) = \frac{\sqrt{\frac{n-2}{n-1}^2 + 1} - 1}{\sqrt{\frac{n-2}{n-1}^2 + 1} + 1} .$$

If $K(n) < \dfrac{\lambda_0}{\lambda_1}$, then (1) - (3) is regular.

This result was established by A.I.Koshelev who also proved its sharpness ([5], [6]). Another proof was given by J.Nečas [7] .

The algebraic condition says that the system in question is not far from the diagonal system with Laplacians. It is easy to see that in case of n=2 it gives another proof of the regularity - in this case K(2) = 0 such that the ratio $\lambda_0/\lambda_1$ is not submitted to any restriction.

At the same time the effort of many mathematicians was directed to make clear the situation in the case $m > 1$ and $n > 2$. The whole collection of examples was constructed to prove that the regularity, in general, doesn't take place. The examples are constructed (as usual) for $m = n \geqq 3$, with a weak solution $u(x) = x/|x|$. (The first one belongs to E.De Giorgi, [8].)

Consider now the quasilinear elliptic system with bounded and continuous coefficients of the form

$$(6) \qquad D_\alpha\left(A_{ij}^{\alpha\beta}(u) D_\beta u^j\right) = 0, \quad i=1,\dots,m,$$

$$(7) \qquad A_{ij}^{\alpha\beta} \in C(R^m), \text{ bounded},$$

$$(8) \qquad A_{ij}^{\alpha\beta}(u)\, \xi_\alpha^i \xi_\beta^j \geqq \lambda |\xi|^2, \quad (\lambda > 0).$$

Recall here again what is known on the regularity: Together with the cases m= 1, n= 2 and the algebraic condition there is a very important class of partial regularity results. They were obtained and further improved by C.B.Morrey [9], E.Giusti, M.Giaquinta, S.Campanato, I.V.Skrypnik, S.N.Kruzhkov and many others. A typical representative is the following

THEOREM. For each weak solution u of (6)-(8) there is a set $\Omega_0 \subset \Omega$, $\Omega_0$ -open, such that

(I) $u \in C^{0,\alpha}_{loc}(\Omega_0)$,

(II) $\Omega \setminus \Omega_0$ (so called singular set) is small.

(In particular, $H_{n-2}(\Omega \setminus \Omega_0) = 0$, where $H_\gamma$ is for $\gamma$-dimensional Hausdorff measure.)

Observe that the above theorem gives a new proof of the regularity if n= 2. Really, in this case $H_0(\Omega \setminus \Omega_0) = 0$, but $H_0(M)$ is equal to card M, so that each singular set is empty. In other words, u is locally Hölder continuous on the whole $\Omega$ for each weak solution u of (6)-(8).

The fact that for m $\geq$2 and n $\geq$3 the system (6)-(8) can be non-regular was demonstrated by many examples. (E.Giusti,M.Miranda [10], V.G.Maz'ja [11], J.Nečas,O.John,J.Stará [12].)

Remark that this introductory part is far from being complete. For more detailed information one can consult [13], [14], [15], [16].

2.NOWHERE CONTINUOUS SOLUTION OF LINEAR ELLIPTIC SYSTEM. Come back to the system (1) - (3). We wanted to answer the following QUESTION 1: Does something like partial regularity takes place in case of linear elliptic systems with bounded measurable coefficients

The first contribution belongs to J.Souček [17] who has constructed to the dense countable subset M of $R^3$ the system (1)-(3) with a weak solution u the singular set of which is just the set M. So the singular set is not necessarily closed.

To expose the following result, we recall that the set $F \subset R^n$ is said to be an $F_\sigma$ -set if F can be written as a union of a countable class of closed subsets of $R^n$. Further, we say that the function v is essentially discontinuous in $x^0$ if

$$\operatorname{oscess} v(x^0) = \inf_{\varepsilon > 0} \inf_{\mu(m)=0} \sup\{|v(x) - v(y)|;\ x,y \in B(x^0,\varepsilon) \setminus m\} > 0.$$

If oscess $v(x^0) = 0$ we say that v is essentially continuous in $x^0$.

In the joint paper of J.Malý,J.Stará and O.John [18] we have proved the following

THEOREM. Let n= m$\geq$ 3. For each $\delta > 0$ and for any $F_\sigma$ -set $F \subset R^n$ there exists a system (1)-(3) and its weak solution u such that

(i) u is essentially discontinuous in each point $x \in F$,

(ii) u is essentially continuous in every $x \in R^n \setminus F$,

(iii) The coefficients of the system satisfy the inequality [4] in a way that $K(n) > \lambda_0/\lambda_1 > K(n) - \delta$, where $K(n)$ is a number given by (5).

Remarks: Firstly, we have proved that any subset F of $F_\sigma$- type in $R^n$ ($n \geq 3$) can be a singular set of a weak solution of an elliptic system with bounded measurable coefficients. As the case $F = R^n$ is not excluded, we can really construct nowhere continuous weak solution.

Secondly, we can see that the vicinity to the algebraic condition does not control neither the structure nor the magnitude of singular sets of weak solutions of the elliptic systems with bounded measurable coefficients.

3.NONISOLATED SINGULARITY IN CASE OF QUASILINEAR SYSTEM (6)- (8).

Recall that the partial regularity result says that singular set of each weak solution u of (6)- (8) is small ($H_{n-2}(\Omega \setminus \Omega_0) = 0$) and closed. Unfortunatelly, a little is known (with except of some particular cases) about the structure of singular sets.

Using 3-dimensional example of non-regular system (6)- (8) with a weak solution $x/|x|$ we can easily construct the system (for $n \geq 4$) with a weak solution whose singular set is a hyperplane $\{x;\ x_1 = x_2 = x_3 = 0\}$. So, for $n \geq 4$, the singular set is not inevitably isolated. Until 1988 remained open the following

QUESTION 2: Is a singular set of a weak solution of (6)- (8) always isolated in a case $n = 3$?

The answer is positive in some particular cases (see e.g. M.Giaquinta, E.Giusti [19]) but is negative in general. It was established by J.Malý [20] who proved the following

THEOREM. There is a system (6)- (8) with $n = 3$ and $m = 6$ for which the weak solution u and the sequence $\{z_k\} \subset R^3$, $z_k \neq 0$, $\lim_{k\to\infty} z_k = 0$ exist such that $\{z_k\} \cup \{0\}$ is the singular set of u.

4.PARABOLIC SYSTEMS. To our previous notations we add $[t,x] = [t,x_1,\dots,x_n]$ for a parabolic point moving in the parabolic cylinder $Q = (0,T) \times B$, where B is a unit ball in $R^n$ with its center at the origin. Denote as $\Gamma$ the parabolic boundary of Q, i.e., $\Gamma = \{0\} \times \overline{B} \cup (0,T) \times \partial B$.

For the linear parabolic system with bounded measurable coefficients

(9) $$\frac{\partial u^i}{\partial t} - D_\alpha\left(A^{\alpha\beta}_{ij}(t,x)\ D_\beta u^j\right) = 0,\ i = 1,\dots,m,$$

(10) $A^{\alpha\beta}_{ij} \in L_\infty(Q)$,

(11) $A^{\alpha\beta}_{ij}(t,x)\ \xi^i_\alpha\ \xi^j_\beta \geq \lambda|\xi|^2$, $(\lambda > 0)$

consider initial-boundary value problem with the condition

(12) $u = u_0$ on $\Gamma$, $u_0$ is a given function.

Denote further $W = \{u \in L_2(Q;R^m)\ ;\ D_\alpha u \in L_2(Q;R^m), \alpha=1,\dots,n\}$.

The vector function $u$ is said to be a weak solution of the problem (9)-(12) if $u$ belongs to W, u is bounded, $u(t,.) = u_0(t,.)$ on $\partial B$ in a sense of traces for almost all $t \in [0,T]$ and the following integral identity is satisfied for all functions $\psi$ infinitely smooth in $\overline{Q}$ with their support in $Q \cup \{0\} \times B$:

$$\int_Q \left[u^i \frac{\partial \psi^i}{\partial t} - A_{ij}^{\alpha\beta}(t,x) D_\beta u^j\, D_\alpha \psi^i\right] dx\, dt = - \int_B u^i \psi^i(0,x)\, dx.$$

Observe that the heat operator (the case m=1, $A_{11}^{\alpha\beta} = \delta_{\alpha\beta}$) has the regularizing effect at any finite time $t > 0$. Does it remain true also for the systems? Take m= n= 3. Let $A_{ij}^{\alpha\beta}(x)$ be the coefficients of the linear elliptic system with a weak solution $w(x) = x/|x|$. The function $u(t,x) = w(x)$ is a stationary solution of initial-boundary value problem (9)-(12) with $A_{ij}^{\alpha\beta}(t,x) = A_{ij}^{\alpha\beta}(x)$ and $u_0(t,x) = x/|x|$. So we have the example of discontinuous weak solution with the discontinuity starting on $\Gamma$ (at the origin) and moving forward along the t-axis.

Now we can ask the following QUESTION 3: Can some weak solution of the problem (9)-(12) start as a smooth function and develop the discontinuity in some moment $t > 0$?

The first example giving the affirmative answer is due to M.Struwe [21]. He considered the diagonal system

(13) $$\frac{\partial u^i}{\partial t} - D_\alpha(A^{\alpha\beta}(t,x) D_\beta u^i) = f_i(t,x,u,Du), \quad i=1,\dots,m,$$

for which, together with

(14) $A^{\alpha\beta} \in L_\infty(Q)$,

(15) $A^{\alpha\beta} \xi_\alpha \xi_\beta \geqq \lambda |\xi|^2$, $(\lambda > 0)$

the quadratic growth condition

(16) $|f(t,x,u,p)| \leqq a\, |p|^2 + b$, (a,b -positive)

is satisfied.

Let $u$ be a weak solution of the system (13)- (16); it is known that

(17) $a\,\lambda^{-1}\,\|u\|_{L_\infty(Q)} < 1$ **implies** that $u \in C^{0,\mu}_{loc}(Q)$.

Struwe constructed his system to the following weak solution: $u(t,x) = x/|x|$ if $t \geqq 1$, $u(t,x) = (x/|x|)*G$ if $t<1$, where $G$ is a fundamental solution of the reverse heat equation $\partial w/\partial t + \Delta w = 0$ for $t < 1$. In his example the condition (17) is strongly violated. At the same time he conjectured that similar example could be constructed for any situation $1 < a\,\lambda^{-1}\|u\|_{L_\infty(Q)} < 1 + \varepsilon$ $(\varepsilon > 0)$.

In the paper [22] we have proved following results:

THEOREM. (i) There is a system (9) and a Lipschitz continuous function $u_0$ on $\Gamma$ such that the problem (9)- (12) has a weak solution $u$ regular in $\overline{Q} \setminus r$ and discontinuous on $r$, where $r = \{[t,x];\ x=0,\ t\geqq 1\}$.

(ii) There is a quasilinear system

$$(9^*)\quad \frac{\partial u^i}{\partial t} - D_\alpha\left(A^{\alpha\beta}_{ij}(u)D_\beta u^j\right) = 0,\ i = 1,\dots,m,$$

(10*) $A^{\alpha\beta}_{ij}$ bounded and continuous on $R^m$ and the ellipticity condition is satisfied

and a Lipschitz continuous function $u_0$ on $\Gamma$ such that the initial-boundary value problem (9*), (10*), (12) has a weak solution with the same property as in (i).

(iii) For any $\varepsilon > 0$ there exists a system (13) with a weak solution $u$ as in the assertion (i) and such that

$$1 < a\,\lambda^{-1}\,\|u\|_{L_\infty(Q)} < 1.25 + \varepsilon\ .$$

Remarks. Firstly, calculating more carefully as in [22], we can get the similar example as in the assertion (i) of our theorem with the estimate (4) for the coefficients in a way that $K(n) > \lambda_0/\lambda_1 > K(n) - \varepsilon\ (\varepsilon>0$, arbitrary$)$, where $K(n)$ is defined by (5).

Secondly, our discontinuous solution is based substantially on elliptic non-regular example. So it does not cover the situation for $n = 2$. Recently, J.Nečas and V.Šverák have proved $C^{1,\mu}$- regularity (for $n \leqq 2$) for the system

$$\frac{\partial u^i}{\partial t} - D_\alpha\left(A^\alpha_i\,(Du)\right) = 0,\quad i = 1,\dots,m,\quad A^\alpha_i \in C^1.$$

5.AN IDEA OF CONSTRUCTING EXAMPLES. ($m = n \geqq 3$). We are going to show briefly how our parabolic example was constructed. Observing the important role of the parabolic variable $\xi = |x|/(2\sqrt{1-t})$ we take the desired solution in the form

$$\bar{u}(t,x) = \begin{cases} \frac{x}{|x|} & \text{if } t \geq 1 \text{ and } x \neq 0, \\ 0 & \text{if } x = 0 \\ \frac{x}{|x|} \cdot \varphi(\xi) & \text{if } t < 1 \text{ and } x \neq 0. \end{cases}$$

The scalar function $\varphi$ is to be found in a way that $\lim_{\xi \to 0+} \varphi(\xi) = 0$, $\lim_{\xi \to +\infty} \varphi(\xi) = 1$.

As $\varphi$ is constant on the parabolas $\xi = c$, the different values are brought to the point $[1,0]$. So at this point the function u develops the discontinuity which runs forward along the t-axis. The careful analysis of Struwe´s example led to the choice $\varphi(\xi) = 2E(\xi) - F(\xi)$, where

$$E(\xi) = \pi^{(-1/2)} \int_0^{\xi} e^{-\tau^2}\, d\tau \quad \text{and} \quad F(\xi) = \xi^{-2}\left[E(\xi) - \pi^{(-1/2)}\, \xi\, e^{-\xi^2}\right].$$

Now we look for a system (9) with a solution $\bar{u}$. Using the method of J.Souček and M.Giaquinta, put

$$A_{ij}^{\alpha\beta}(t,x) = \delta^{\alpha\beta}\delta_{ij} + \frac{d_i^{\alpha}\, d_j^{\beta}}{(d, D\bar{u})}, \quad d_i^{\alpha} = b_i^{\alpha} - D_{\alpha}\bar{u}^i.$$

Substituting this special form of coefficients into (9) we can see that the vector field $\{b_i^{\alpha}\}$ must satisfy the conditions

$$(18) \quad \frac{\partial \bar{u}^i}{\partial t} = D_{\alpha} b_i^{\alpha}, \quad i = 1,\dots,m.$$

To obtain the coefficients bounded we need that the singularities of $\{d_i^{\alpha}\}$ are of the same type as those of $\{D_{\alpha}u^i\}$. Calculating

$$D_{\alpha}\bar{u}^i = \frac{1}{|x|}\left[\delta_{\alpha i}(\tilde{\alpha}E + \tilde{\beta}F) + \frac{x_{\alpha}x_i}{|x|^2}(\tilde{\gamma}E + \tilde{\delta}F)\right]$$

($\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ - real coefficients) we put

$$b_i^{\alpha} = \frac{1}{|x|}\left[\delta_{\alpha i}(aE + gF) + \frac{x_{\alpha}x_i}{|x|^2}(cE + dF)\right].$$

From (18) we obtain three linear algebraic equations for four parameters a, g, c, d. The remaining free parameter is chosen in a way that the ellipticity condition is guaranteed.

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