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AN ELECTROMAGNETIC FREE-BOUNDARY PROBLEM

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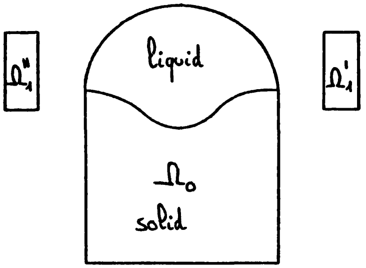

Figure 1

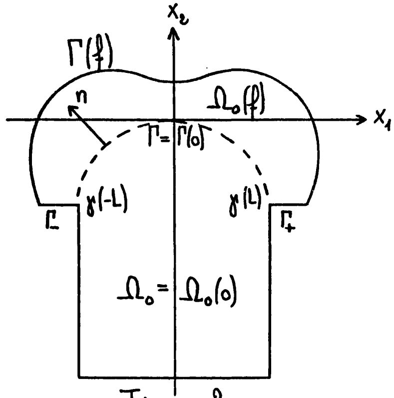

Figure 2

A section of the electromagnetic casting (EMC) device is schematically represented in Figure 1. Ω_0 represents the ingot (aluminium) with a solid part and a liquid part. An alternating current run along the inductors Ω_1', Ω_1''; it induces an electromagnetic field which, in its turn, produces Laplace forces in the ingot; these forces compensate the action of the gravity and maintain the liquid metal in equilibrium. The interface liquid-air is the main unknown of the problem. The ingot is supposed to be infinitely long so that the free boundary problem is two dimensional.

In Figure 2, Ω_0 is a given approximation of the section of this ingot at equilibrium; $\partial\Omega_0$ is smooth except at the two corners at the bottom. $\Gamma \subset \partial\Omega_0$ (dotted line) is the interface and we suppose that $0 \in \Gamma$. $\Omega = \Omega_0 \cup \Omega_1$, where $\Omega_1 = \Omega_1' \cup \Omega_1''$, is symmetric with respect to the x_2-axis. Γ admits the parametrization $x = (x_1,x_2) = \gamma(\xi)$, $-L \le \xi \le L$, where ξ is the arc length parameter. The unit exterior normal is denoted by $n(\xi) = (n_1(\xi), n_2(\xi))$. We introduce, in the neighbourhood of Γ, the orthogonal curvilinear system of coordinates (ξ,η) defined by the relation $x = \gamma(\xi) + \eta\, n(\xi)$. Let $W = \{f \in C^0[-L,L] \mid \|f\| < \varepsilon\}$ where $\|\cdot\|$ is the uniform norm and $\varepsilon > 0$ is chosen small enough in accordance to the geometry. For $f \in W$, let $\Gamma(f)$ be defined by the parametrization $x = \gamma(\xi) + f(\xi)\,\eta(\xi)$, $-L \le \xi \le L$. $\Omega_0(f)$ is then defined as the domain with boundary $(\partial\Omega_0 - \Gamma) \cup \Gamma(f) \cup \Gamma_+ \cup \Gamma_-$ where $\Gamma_\pm$ are the two horizontal segments described in Figure 2. Finally we set $\Omega(f) = \Omega_0(f) \cup \Omega_1$.

We rely on [1], [2], [3] for a justification of the mathematical model. Let $W_0^1(\mathbb{R}^2)$ be the completion of $C^\infty(\mathbb{R}^2)$ (complex functions) for the norm $\|v\|_{W_0^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\nabla v|^2 + \int_{\Omega_0} |v|^2$ (see [1] for details). For $f \in W$, we define $a(f) : W_0^1(\mathbb{R}^2) \times W_0^1(\mathbb{R}^2) \to \mathbb{C}$, $b(f) : W_0^1(\mathbb{R}^2) \to \mathbb{C}$:

$$a(f)(u,v) = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \bar{v} - 2i\,\alpha^2 \int_{\Omega} u\bar{v}, \quad b(f)(v) = 2i\,\alpha^2 \int_{\Omega_1} d\bar{v}. \tag{1}$$

Here α is a (large) real constant depending on the conductibility of the metal and on the frequency of the currents. i is the complex unit, $\bar{v}$ is the complex conjugate of v. $d : \Omega_1 \to \mathbb{R}$ is a real odd function with respect to x_1, the restrictions of which on Ω_1' and Ω_1'' are constant; d is related to the intensity of the current running in the inductors.

The proofs of the results stated in this paper can be found in [2].

Proposition 1 . For $f \in W$, there exists a unique $\varphi(f) \in W_0^1(\mathbb{R}^2)$ such that

$$a(f)(\varphi(f),v) = b(f)(v), \qquad \forall\, v \in W_0^1(\mathbb{R}^2). \tag{2}$$

$\varphi(f)$ is a potential the bidimensional curl of which is the magnetic field. In the neighbourhood of Γ, $\varphi(f)$ will be considered as a function of the curvilinear coordinates (ξ,η). For $f \in W$, $H(f)$ is the function defined on $\Gamma(f)$ by

$$H(f)(\xi) = \frac{1}{2}\, C_m\, |\varphi(f)\,(\xi,f(\xi))|^2 + C_g\, x_2(\xi,f(\xi)). \tag{3}$$

The first term is (2) represents an approximation of the "magnetic pressure" on $\Gamma(f)$; the second term gives the pressure due to the gravity; C_m and C_g are positive constants. The equilibrium is obtained when the total pressure $H(f)$ is constant along $\Gamma(f)$. Due to an industrial constraint explained in [2], this constant vanishes in our case. Consequently, our problem is to find $f \in W$ such that $H(f) = 0$. Because of the complexity of the geometry, it seems to us not realistic to prove a mathematical existence theorem. We however have the following results.

We consider a, φ and H as function defined on W with values in the sesquilinear forms on $(W_0^1(\mathbb{R}^2))^2$ forms, in $W_0^1(\mathbb{R}^2)$ and in $C^0[-L,L]$, respectively. Let D and ∂_η denote the Fréchet derivative with respect to f and the partial derivative with respect to η in the coordinate system (ξ,η). $J(\xi,\eta)$ will denote the jacobian of the transformation $(\xi,\eta) \to (x_1,x_2)$.

Proposition 2

a) For any bounded domain $\Lambda \subset \mathbb{R}^2$ and any $2 \le p \le \infty$, one has

$$\varphi \in C^1(W,W_0^1(\mathbb{R}^2)) \cap C^1(W,W^{1,p}(\Lambda)) \cap C^0(W,W^{2,p}(\Lambda)).$$

b) For $h \in C^0[-L,L]$, $D\varphi[h]$ is characterized by the variational equality

$$a(f)\,(D\varphi(f)[h],v) = -\,Da(f)[h](\varphi(f),v), \qquad \forall\, v \in W_0^1(\mathbb{R}^2), \tag{4}$$

where $Da(f)[h](u,v) = -\,2i\,\alpha^2 \int_{-L}^{L} J(\xi,f(\xi))\cdot h(\xi)\cdot u(\xi,f(\xi))\cdot \bar{v}(\xi,f(\xi))\, d\xi.$

Proposition 3

a) There exists $0 < \mu < 1$ such that $H \in C^{1,\mu}(W, C^0[-L,L])$.

b) $DH(f)\,[h]\,(\xi) = C_m \operatorname{Re}\{\, \bar{\varphi}(f)\,(\xi,f(\xi))\cdot(D\varphi\,(f)\,[h]\,(\xi,f(\xi)) + \partial_\eta\varphi(f)\,(\xi,f(\xi))\cdot h(\xi))\,\} + C_g h(\xi)\,\eta_2(\xi).$

It can be shown that $DH(f)$ can be expressed as the sum of a compact operator and a multiplication operator of the form $r(\xi)h(\xi)$, where $r \in C^0[-L,L]$. A physical analysis shows that it is realistic to suppose that $r(\xi) \neq 0$, $\forall\, \xi \in [-L,L]$. Under this hypothesis $DH(f)$ is a Fredholm operator.

Proposition 3 suggests the use Newton's method for solving the equation $H(f) = 0$. Numerically, this implies the discretization of (2) and (4); this has been realized efficiently in [2] with boundary elements; 90 % of the time necessary to compute one iteration is used for determining φ in (2). Numerical experiments for concrete situations show a very fast convergence; three Newton iterations are generally sufficient.

For more details and other approaches to the EMC problem, see [2], [3].

References

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