Lothar Collatz

Numerical calculations for some types of boundary value problems with unbounded domains

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NUMERICAL CALCULATIONS FOR SOME TYPES OF BOUNDARY VALUE PROBLEMS WITH UNBOUNDED DOMAINS

COLLATZ L., HAMBURG, FRG

Abstract: An important but often difficult problem for the numerical analysis is the question for the accuracy of an approximate solution of an initial or boundary value problem; this question can be answered for some simple models with aid of approximation and optimization methods. It is useful for application methods to generalize the theory and the H-sets to more general classes of approximating functions, for instance of rational, algebraic and other functions.

1. Introduction and H-sets. An important task for the numerical analysis is the determination of bounds for the error of an approximate solution; or in practice: The decision how many of the digits of the computer-result can be guaranteed. These error bounds or "exact inclusions" (that means guaranteeable) help to compare different numerical methods and to see how accurate they are. For brevity we restrict ourselves on some simple numerical examples.

The theory of H-sets permits often to characterize a "best"-Chebyshev-Approximation and to see which accuracy is possible with a prescribed number p of parameters in a given class $W=W(x,a)=\{w(x_1,\dots,x_m,a_1,\dots a_p)\}$ of continuous functions w, which are used for approximation of a in a domain B given continuous function f in the maximum norm. For a more detailed description see f. i. Meinardus |67|, Collatz-Krabs |73|, Werner |62|, Hämmerlin-Hoffmann |89|, Engeln-Müllges |88| a.o. . A H-set is the union of two point sets M_1, M_2 in B with the property: There exists no pair of two elements $w, \tilde{w}$ of W with

$$w - \tilde{w} > 0 \text{ on } M_1, \quad w - \tilde{w} < 0 \text{ on } M_2. \tag{1.1}$$

Let be $w \in W$ with the errorfunction $\varepsilon = \tilde{w} - f$. If $\varepsilon > 0$ on M_1, $\varepsilon < 0$ on M_2, and ρ the minimal distance: (1.2) $\rho = \inf_{w \in W} ||w-f||$, then we have the inclusion theorem for ρ:

$$\mu = \min_{x \in H} |\varepsilon(x)| \le \rho \le M = ||\varepsilon||. \tag{1.3}$$

If $\mu = M$, then $\tilde{w}$ is a best approximation.

Important remark: Let $\psi(z)$ be a monotonically increasing (or monotonically decreasing) function of the real variable z; if a set Q is a H-set for the class $W=\{w(x,a)\}$, then Q is also a H-set for the class $W^*=\{\psi(w(x,a,))\}$.

H-sets are often considered for polynomials P(x) in one and more dimensions and all these H-sets are also H-sets f.i. for $\frac{1}{P(x)}$, $\sqrt{P(x)}$, $\frac{1}{\sqrt{P(x)}}$, $\sqrt[3]{P(x)}$, $|P(x)|^q$ for $q > 0$, $\ln P(x)$, $e^{P(x)}$ a.o.

Example: Approximate $f(\alpha) = \sqrt{2-x}$ by $w(x) = \frac{a_1}{\sqrt{a_2+x}}$ in the interval $[0,1]$.

Any three abscissae ξ,η,ρ with $0 \le \xi < \eta < \rho \le 1$ with $\xi,\rho \in M_1$, $\eta \in M_2$ are a H-set; in the example shows the symbolic scetch the best approximation, Fig. 1, with $\rho=0.04$.

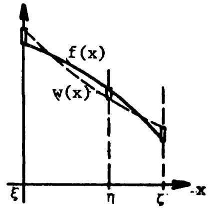


Fig.1

The theory is applicable to rational, algebraic and other approximation; we show that these approximations occur in many applications.

2. Rational approximation

A) Unbounded domains. Ideal flow of a liquid over ground, described in a x-y-plane by a function $y=\psi(x)$, for instance $\psi(x)=\frac{x^2}{1+x^4}$, Fig. 2.

The streamlines may be $\hat{u}(x,y)=$const. We have for $u = y-\hat{u}$

$$(2.1)\begin{cases}\Delta u=0 \text{ in } B = \{(x,y),\ |x|<\infty,\ y<\psi(x)\}\\ u(x,\psi(x)) = \psi(x) \text{ on } \Gamma=\{(x,y),y=\psi(x)\}\\ u=0 \text{ "at infinity"}\end{cases}$$

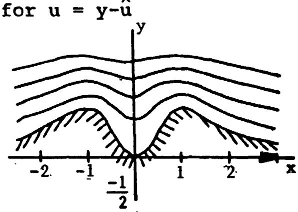


Fig.2

We take as approximate solution v(x,y) the rational function

$$(2.2)\quad v(x,y) = \sum_{p=1}^{m} a_p\left(\frac{y-\eta_p}{(x-\xi_p)^2+(y-\eta_p)^2} + \frac{y-\eta_p}{(x+\xi_p)^2+(y-\eta_p)^2}\right) +$$

$$+ \sum_{p=m+1}^{n} a_p\left(\frac{2(x-\xi_p)(y-\eta_p)}{((x-\xi_p)^2+(y-\eta_p)^2)^2} - \frac{2(x+\xi_p)(y-\eta_p)}{((x+\xi_p)^2+(y-\eta_p)^2)^2}\right)$$

Here holds the monotonicity (compare f.i. Collatz |52|,|66|,|81|, Bohl |74|, Schröder |80| a.o.)

$$(2.3)\quad \text{From}\begin{cases}T\phi = \begin{pmatrix}-\Delta\phi \text{ in } B\\ \phi \text{ on } T_1\end{pmatrix} \ge 0\\ \text{and}\quad \phi=0 \text{ at infinity}\end{cases}\ \text{follows } \phi \ge 0 \text{ in } B \cup \partial B.$$

The sign $\ge$ holds for every component and pointwise in the classical ordering for real numbers. Taking $\varepsilon=v-u$ as ϕ, one can get an upper bound for u.

I thank Mr. M. Dellnitz for numerical calculation. He got for m=2, n=3 the values for the parameters:

p	α_p	ξ_p	η_p
1	0.075 200	0.989 617	0.118 975
2	0.334 259	0.159 502	-1.345 361
3	0.251 144	0.673 395	-0.780 159

and the guaranteed error bound

$$|v-u| \leq 0.00418 \quad \text{in} \quad B.$$

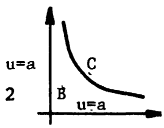
Adding the higher singularity for p=3 increased the accuracy.

B) Outer space problems $\psi(x,y,z)$ may be a continuous function: $\psi=0$ may describe the surface ∂B of a connected closed bounded domain B and $\psi>0$ describe the "outer space" R^3 B. We suppose $\psi(0,0,0)<0$. Problem for u:

$$(2.4) \quad \begin{cases} \Delta u = 0 \text{ in } R^3\setminus B \text{ and } u = f(x,y,z) \text{ on } \partial B \text{ (for } \psi=0) \\ \lim\limits_{r\to\infty} u=0 \text{ (with } r^2=x^2+z^2) \end{cases}$$

f(x,y,z) is given. We approximate u by $v(x,y,z) = \sum\limits_{j=1}^{p} a_j r_j$, where r_j is the euclidean distance from the point (x,y,z) to a chosen fixed inner point p_j of B.

C) Blow-up-phenomena (Friedman 1982-1989 a.o.)



$$(2.5) \quad \text{Differential equation } \frac{\partial^2 u}{\partial x \partial y} = p(x,y)+q(x,y)\,[u(x,y)]^2$$

$$\text{in } B = \{(x,y,);\ x>0,\ y>0\}.$$

Fig.3

Initial conditions $u(x,0)=u(0,y)=a=\text{const} > 0$; p(x,y) and q(x,y) are given continuous functions with $p\geq 0$, $q>0$ in B, q in x and y monotonically not decreasing. Then exists a curve C in B with $\lim u(x,y)=\infty$, if x,y approaches a point of C from smaller values of x,y, Fig. 3.
Example: $\partial^2 u/(\partial x \partial y)=1+u^2$; a=1; approximate rational solution for instance $v(x,y)=(1+a_1xy+a_2x^2y^2)/(1-a_3xy)$.

This phenomenon occurs also for ordinary Differential Equation

$$(2.6) \quad y'(x) = q(x) + \sum_{\nu=0}^{k} p_\nu(x)\ [y(x)]^\nu,\ y(x_0) = y_0, \text{ in } B=\{x; x_0<x<\infty\}$$

3. Algebraic approximation

A) Outer space problem.

We use the notations of Nr. 2, B):

$B = \{(x,y,z),\ r^2=x^2+y^2+z^2>1\}$, Fig. 4.

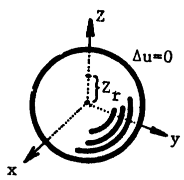


Fig.4

Problem:

$\Delta u = 0$ in B; $u = z^2$ on ∂B; $u = 0$ "at infinity".

Approximation with $\frac{1}{\sqrt{x^2+y^2+(z-z_\nu)^2}}$

I thank Mr. Th. Schiemann for calculation with 3 poles.
He got an approximate solution v with an error bound
$|v-u| \leq 0.000038$.

(more details in Collatz [82]).

B) Blow-up-Problem:

We consider a special case of (2.5): $Ty = y'(x)-x-[y(x)]^3=0, y(0)=1$.
We ask for a value $x=\xi$ with

$$\lim_{x\to\xi} y(x) = \infty,\ y(x)>1 \text{ for } 0<x<\xi.$$

$\phi = (1-cx)^{-1}$ gives $T\phi \leq 0$ for $c=2$; therefore $\xi < \frac{1}{2}$

We cannot reach $T\phi \geq 0$ for $c > 0$; but the algebraic function
$\psi = \frac{1}{\sqrt{1-2ax}}$ gives $T\psi \geq 0$ for $a = 1.085$; therefore $\xi > \frac{1}{2a} = \frac{1}{2.17} = 0.4608$.
There are also many other cases in which algebraic approximation is appropriate.

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