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The accuracy of numerically computed orbits of dynamical systems


Persistent URL: http://dml.cz/dmlcz/702394

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In their papers [2,3] Hammel, Yorke and Grebogi have given a procedure which determines the accuracy of numerically computed orbits of dynamical systems. They apply their procedure to maps which exhibit a large amount of hyperbolicity. However their procedure does not use the hyperbolicity explicitly. In this paper we give a procedure for one-dimensional maps which does use the hyperbolicity explicitly. Unlike the procedure of Hammel et al., our procedure works forward. After \( N \) iterates we can decide whether our theorem applies and, if it does, we can estimate how far the computed orbit is from a true orbit.

Now we state the main theorem. Let \( f : [0,1] \to [0,1] \) be a \( C^2 \) function and let \( \{y_n\}_{n=0}^{N+1} \) be a pseudo-orbit of this map, i.e., \( |y_{n+1} - f(y_n)| \) is small for \( n = 0, 1, \ldots, N \). We define the quantities

\[
\sigma = \sup_{n=0}^{N} \sum_{m=n}^{N} |Df(y_n)^{-1}Df(y_{n+1})^{-1} \ldots Df(y_m)^{-1}|,
\]

which measures the expansiveness of the map, and

\[
\tau = \sup_{n=0}^{N} \sum_{m=n}^{N} |Df(y_n)^{-1}Df(y_{n+1})^{-1} \ldots Df(y_m)^{-1}|y_{m+1} - f(y_m)|.
\]

It turns out that \( \tau \) gives a good measure of how close the pseudo-orbit (of course, our numerically computed orbits will be pseudo-orbits) is to a true orbit.

**THEOREM.** Let \( f : [0,1] \to [0,1] \) be a \( C^2 \) function with

\[
M = \sup \{|D^2f(x)| : 0 \leq x \leq 1\}.
\]

Let \( \{y_n\}_{n=0}^{N+1} \) be a pseudo-orbit of \( f \) such that

\[
2M\tau \leq 1.
\]

Then there is an exact orbit \( \{x_n\}_{n=0}^{N} \) with

\[
(1 + 1/2(1 + \sqrt{1 - 2M\tau}))^{-1} \leq \sup_{n=0}^{N} |x_n - y_n| \leq 2(1 + \sqrt{1 - 2M\tau})^{-1} \tau.
\]

**Outline of proof.** Denote by \( S \) the set of sequences \( x = \{x_n\}_{n=0}^{N} \) with \( |x_n - y_n| \leq \varepsilon \) for \( n = 0, 1, \ldots, N \), where

\[
\varepsilon = 2\tau/(1 + \sqrt{1 - 2M\tau}).
\]

\( S \) is a compact convex subset of \( \mathbb{R}^{N+1} \). We define a mapping \( T \) on \( S \). If \( x \in S \) we define

\[
(Tx)_n = y_n - \sum_{m=n}^{N} Df(y_n)^{-1}Df(y_{n+1})^{-1} \ldots Df(y_m)^{-1}h_m \quad (n = 0, \ldots, N),
\]

where

\[
h_n = f(x_n) - y_{n+1} - Df(y_n)(x_n - y_n).
\]
It turns out that \( T \) is a continuous mapping of \( S \) into itself and so, by Brouwer’s fixed point theorem, has a fixed point \( x = \{x_n\}_{n=0}^N \). This is the exact orbit that we wanted.

Note that the idea of this proof was suggested by the proofs of the shadowing lemma given in Palmer [4] and Chow, Lin and Palmer [1].

**The Method of Computation**

Let \( f : [0,1] \rightarrow [0,1] \) be a \( C^2 \) mapping. Suppose our computer starts with a number \( y_0 \) in \( [0,1] \) and computes an orbit \( \{y_n\}_{n=0}^{N+1} \) of \( f \) in single precision. \( \{y_n\} \) will be, in fact, a pseudo-orbit. To use the theorem we have to find the quantities \( \sigma \) and \( \tau \). For large \( N \) it would not be practical to compute the sums \( \sum_{m=n}^N \). Instead we calculate the quantities

\[
\sigma_p = \sup_{n=0}^N \sum_{m=n}^{\min(n+p,N)} |Df(y_n)^{-1}\ldots Df(y_m)^{-1}|, \\
\tau_p = \sup_{n=0}^N |\sum_{m=n}^{\min(n+p,N)} Df(y_n)^{-1}\ldots Df(y_m)^{-1}[y_{m+1} - f(y_m)]|,
\]

where \( p \) is an integer, \( 0 \leq p \leq N \), such that

\[
\mu_p = \sup_{n=0}^{N-p} |Df(y_n)^{-1}\ldots Df(y_{n+p})^{-1}| < 1.
\]

It turns out that

\[
\sigma \leq (1 - \mu_p)^{-1}\sigma_p, \quad \tau \leq (1 - \mu_p)^{-1}\tau_p.
\]  

(1)

The computation of \( \mu_p, \sigma_p, \tau_p \) is done in double precision. We have fully analyzed the effect of round-off error on these computations. Unless the hyperbolicity if very weak (i.e. \( \sigma \) is large and \( \mu_p < 1 \) only for large \( p \)), it turns out that the effect of round-off error is very slight.

**Example.** We consider the quadratic map \( f(x) = ax(1-x) \) with \( a = 3.8 \). Then \( M = 2a = 7.6 \). The computations were done on an IBM compatible computer using Microsoft Quickbasic. For \( N = 426,000, p = 30 \) and \( y_0 = .3 \), we find that

\[
\mu_p = 2.297433184600331 \times 10^{-3}, \\
\sigma_p = 375.6005726956602, \\
\tau_p = 9.60282364278178 \times 10^{-6}.
\]

Using the inequalities (1) and taking into account the round-off error, we find that

\[
\sigma \leq 376.4658, \quad \tau \leq 9.624939 \times 10^{-6}.
\]

Then \( 2M\sigma \leq 0.5507661 \) and

\[
2\tau/(1 + \sqrt{1 - 2M\sigma}) \leq 9.76125 \times 10^{-6}.
\]

Our theorem enables us to conclude that during 426,000 iterates our computed orbit differs by at most \( 1/10^6 \) from a true orbit. Note that the orbit was computed only in single precision, that is to an approximate accuracy of 7 decimal digits. So over 426,000 iterates we have only lost two digits of accuracy.
References


