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DOMAIN OPTIMIZATION IN AXISYMMETRIC ELLIPTIC PROBLEMS

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One often meets elliptic boundary value problems in 3D-domains, which are generated by the rotation of a bounded plane domain about an axis. Then the natural approach is to use cylindrical coordinates (r,ϑ,z). If the data are axisymmetric, the problem is reduced to the meridional section D.

Let a part $\Gamma(\alpha)$ of the boundary ∂D be optimized, so that a cost functional attains its minimum. We shall consider the State Problem

$$A\,y = f \quad \text{in } D(\alpha), \qquad (\, y = y(r,z)),$$

where A is a linear elliptic operator with two variants of A, namely:

I. $$-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\, a_r \frac{\partial y}{\partial r}) + \frac{\partial}{\partial z}(\, a_z \frac{\partial y}{\partial z}\,)\right],$$

II. Lamé's system of linear elastostatics.

Let us denote $D(\alpha) = \{(r,z) \mid 0 < r < \alpha(z),\ 0 < z < 1\}$, $\Gamma(\alpha)$ the graph of the function α, $\Gamma_1 = \{(r,0),\ 0 < r < \alpha(0)\}$, where α belongs to the following set of admissible functions

$$U_{ad} = \{\alpha \in C^{(0),1}([0,1]) \text{ (i.e. Lipschitz function)},$$
$$0 < \alpha_{min} \leq \alpha(z) \leq \alpha_{max},\quad |d\alpha/dz| \leq C_1,$$
$$\int_0^1 \alpha^2(z)\, dz = C_2\}$$

and α_{min}, α_{max}, C_1, C_2 are given parameters.

We shall use weak formulations of the State Problems. To this end, we introduce weighted Sobolev space $W_{2,r}^{(1)}(D)$ with the norm

$$\left(\int_D \left[u^2 + (\frac{\partial u}{\partial r})^2 + (\frac{\partial u}{\partial z})^2\right] r\, dr\, dz\right)^{1/2} = \|u\|_{1,r,D}$$

and the space of test functions

$$V(D(\alpha)) = \{\, v \in W_{2,r}^{(1)}(D(\alpha)) \mid \gamma v = 0 \quad \text{on } \Gamma_2\,\},$$

where γ is the trace operator. Then the State Problem takes the following form: find $y \in V(D(\alpha))$ such that

(1) $$a(\alpha,y,v) = L(\alpha,v) \quad \forall\ v \in V(D(\alpha)).$$

In **case** I of the single equation

$$a(\alpha,y,v) = \int_{D(\alpha)} \left(a_r \frac{\partial y}{\partial r} \frac{\partial v}{\partial r} + a_z \frac{\partial y}{\partial z} \frac{\partial v}{\partial z}\right) r \, dr \, dz ,$$

$$L(\alpha,v) = \int_{D(\alpha)} f \, v \, r \, dr \, dz + \int_{\Gamma_1(\alpha)} g \, v \, r \, dr .$$

Here the coefficients a_r and a_z are given in the space $L^\infty(\hat{D})$,where $\hat{D} = (0,\delta) \times (0,1), \delta > \alpha_{max}$ and there exists a positive constant a_0 such that $a_r \geq a_0$, $a_z \geq a_0$ holds a.e. in $\hat{D}$. Moreover, $f \in L_{2,r}(\hat{D})$ and $g \in L_{2,r}(\hat{\Gamma}_1)$ are given functions.

In case II of elasticity we formulate the State Problem in terms of the displacement vector $\underline{y} = (u,w)$ and introduce the following space and bilinear form:

$$V(D(\alpha)) = \left\{ (u,w) \,\middle|\, u \in W^{(1)}_{2,r}(D(\alpha)) \cap L_{2,1/r}(D(\alpha)),\ w \in W^{(1)}_{2,r}(D(\alpha)),\ \gamma u = \gamma w = 0 \text{ on } \Gamma_2 \right\},$$

$$a(\alpha,\underline{y},\underline{v}) = \int_{D(\alpha)} \left[\sigma_r(\underline{y})\varepsilon_r(\underline{v}) + \sigma_\vartheta(\underline{y})\varepsilon_\vartheta(\underline{v}) + \sigma_z(\underline{y})\varepsilon_z(\underline{v}) + 2\sigma_{rz}(\underline{y})\varepsilon_{rz}(\underline{v})\right] r \, dr \, dz,$$

where $\underline{v} = (\varrho,\zeta)$ and the strain components are

$$\varepsilon_r(\underline{v}) = \frac{\partial \varrho}{\partial r} , \ \varepsilon_\vartheta(\underline{v}) = \frac{\varrho}{r}, \ \varepsilon_z(\underline{v}) = \frac{\partial \zeta}{\partial z}, \ \varepsilon_{rz}(\underline{v}) = \left(\frac{\partial \varrho}{\partial z} + \frac{\partial \zeta}{\partial r}\right)/2 .$$

The stress components $\sigma_r, \sigma_z, \sigma_\vartheta, \sigma_{rz}$ are given as linear forms in terms of the strain components (by a generalized Hooke's law). The functional $L(\alpha,\underline{v})$ represents a virtual work of external forces

$$L(\alpha,\underline{v}) = \int_{D(\alpha)} [f_r \varrho + f_z \zeta] r \, dr \, dz + \int_{\Gamma_1(\alpha)} [g_r \varrho + g_z \zeta] \ r \, dr ,$$

where $f_r, f_z \in L_{2,r}(\hat{D})$ and $g_r, g_z \in L_{2,r}(\hat{\Gamma}_1)$.

There exists a unique solution $y = y(\alpha)$ of the State Problem (1) for any $\alpha \in U_{ad}$ in both cases I and II.

We consider four different types of the <u>cost functionals</u>:

$$j_1(\alpha,y) = \begin{cases} \int_{D(\alpha)} (y - y_d)^2 \, r \, dr \, dz & \text{(I) } (y_d \text{ given}), \\ \int_{D(\alpha)} (u^2 + w^2) \, r \, dr \, dz & \text{(II)} \end{cases}$$

$$j_2(\alpha,y) = \int_0^1 [y(\alpha(z),z - y_d]^2 \, dz \quad \text{(I)} \quad (y_d \text{ given})$$

$$\int_0^1 [(u(\alpha(z),z) - u_g)^2 + (w(\alpha(z)) - w_g)^2] \, dz \quad \text{(II)}$$

$$j_3(\alpha,y) = a(\alpha,y,y) \quad \text{(I,II)}$$

$$j_4(\alpha,y) = \int_{D(\alpha)} \left[(a_r \frac{\partial y}{\partial r} - K_1)^2 + (a_z \frac{\partial y}{\partial z} - K_2)^2 \right] r\, dr\, dz \qquad (I)$$

$$\int_{D(\alpha)} \mu^2 \left[\varepsilon_r^{\,2}(y) + \varepsilon_\vartheta^2(y) + \varepsilon_z^{\,2}(y) + 2\varepsilon_{rz}^{\,2}(y) - \frac{1}{3}(\varepsilon_r(y) + \varepsilon_\vartheta(y) + \varepsilon_z(y))^2 \right] r\, dr\, dz \qquad (II) .$$

Note that $j_3(\alpha,y(\alpha)) = L(\alpha,y(\alpha))$ (i.e.,so called compliance) and the quadratic form in j_4 for the case II is proportional to the square of the von Mises function.

Now we may formulate the Domain Optimization Problems (I or II)

$$(P_i) \qquad \alpha^0 = \arg\min_{\alpha \in U_{ad}} j_i(\alpha,y(\alpha)) , \quad i \in \{1,2,3,4\} .$$

Theorem. There exists at least one solution of the problem (P_i) for all $i \in \{1,2,3,4\}$.

To define approximate solutions of (P_i),we employ standard finite element spaces V_h, consisting of piecewise linear functions on triangulations $\mathcal{T}_h(\alpha_h)$, where α_h is a piecewise linear approximation belonging to U_{ad}.

Instead of the problem (1) we consider the Approximate State Problems: find $y_h = y_h(\alpha_h) \in V_h$ such that

$$a(\alpha_h, y_h, v_h) = L_h(\alpha_h, v_h) \quad \forall\, v_h \in V_h ,$$

where $L_h(\alpha_h,v_h)$ is a suitable approximation of $L(\alpha_h,v_h)$ by means of a simple numerical integration formula. We arrive at the following Approximate Domain Optimization Problems: find

$$(P_h)_i \qquad \alpha_h^0 = \arg\min_{\alpha_h \in U_{ad}^h} j_i(\alpha_h, y_h(\alpha_h)) \qquad i \in \{1,2,3,4\} .$$

In papers[1],[2] the following results were proved. If the data f,g or f_r,f_z,g_r,g_z are regular enough, then every sequence $\{\alpha_h\}$, $h \to 0$, of solutions of the problem $(P_h)_i$, $i \in \{1,2,3\}$, contains a subsequence,converging to a function α uniformly,which appears to be a solution of the problem (P_i).

Moreover,the approximate solutions $y_h(\alpha_h)$ converge also to the exact solution $y(\alpha)$ in a certain sense.

In the end, a dual finite element approximation of the State Problem has been employed in Case I for a generalized cost functional $j_4(\alpha,y)$, considering slightly different configuration of boundary conditions. The details are to be published in the paper [3]. Here we have used the finite element model and the error analysis presented in the paper [4].

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