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CLIFFORD ALGEBRAS AND THE DOUBLE-LAYER POTENTIAL OPERATOR

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1. Introduction

In recent years Clifford algebras have been used in a number of parts of analysis and differential geometry. They have been used for some time in mathematical physics. In sections 2 and 3 I shall outline some basic concepts. This material is essentially taken from [1]. In the remaining sections I shall indicate why Clifford algebras are relevant to proving the L_2-boundedness of the double-layer potential operator on Lipschitz surfaces. This is an extension of work of Coifman and Murray [4]. Details will appear in the proceedings of the Banach Center [3].

I would like to thank the organizers of the Spring School for their exceptionally kind hospitality.

2. Clifford algebras

Consider $\mathbb{R}^{n+1}$ with the standard basis, written here as $e_0, e_1, \dots, e_n$. We regard $\mathbb{R}^n$ as the subspace generated by $e_1, \dots, e_n$. We define a real 2^n-dimensional vector space, $\mathbb{R}_{(n)}$, as being generated by $\{e_S \mid S \subset \{1\dots n\}\}$. We regard

$$\mathbb{R}^{n+1} \subset \mathbb{R}_{(n)} \quad \text{via the embedding} \quad e_0 \to e_\emptyset, \quad e_j \to e_{\{j\}}$$
$$j = 1, \dots, n .$$

We make $\mathbb{R}_{(n)}$ an algebra by defining

$$e_0 = 1$$

$$e_j^2 = -e_0 = -1, \quad j = 1\dots n$$

$$e_j e_k = -e_k e_j = e_{\{j,k\}} \quad 1 \leq j < k \leq n$$

and more generally

$$e_{j_1} e_{j_2} \dots e_{j_s} = e_S, \quad 1 \leq j_1 < j_2 < \dots < j_s \leq n,$$
$$S = \{j_1 \dots j_s\},$$

and for $\lambda, \mu \in \mathbb{R}_{(n)}$,

$$\lambda = \sum_S \lambda_S e_S \; , \quad \mu = \sum_T \mu_T e_T$$

we have

$$\lambda\mu = \sum_{S,T} \lambda_S \mu_T e_S e_T \; .$$

If $n = 0$, then $\mathbb{R}^{0+1} = \mathbb{R}_{(1)} = \mathbb{R}$; if $n = 1$, then $\mathbb{R}^{1+1} = \mathbb{R}_{(2)} = \mathbb{C}$, the complex numbers; if $n = 2$, then $\mathbb{R}^{2+1} \subsetneq \mathbb{R}_{(2)}$, the quaternions. If $n \geq 2$ then $\mathbb{R}_{(n)}$ is not commutative. If $n \geq 3$, then there exist non-zero elements $\lambda, \mu \in \mathbb{R}_{(n)}$ such that $\lambda\mu = 0$. Our interest though is really in elements of $\mathbb{R}^{n+1}$, in which case the following theorem applies. The conjugate of the element

$$x = x_0 e_0 + x_1 e_1 + \ldots + x_n e_n$$

is

$$\bar{x} = x_0 e_0 - x_1 e_1 - \ldots - x_n e_n \; .$$

<u>PROPOSITION 1</u>. *If* $x, y \in \mathbb{R}^{n+1} \subset \mathbb{R}_{(n)}$, *then*

(i) $x\bar{y} = \langle x,y\rangle + \sum_{0 \leq j < k \leq n} (x_k y_j - x_j y_k) e_j e_k$,

(ii) $x\bar{x} = \bar{x}x = |x|^2$,

(iii) *if* $x \neq 0$, *then* x *has an inverse,* $x^{-1} = |x|^{-2}\,\bar{x}$.

The existence of an inverse x^{-1} of non-zero elements $x \in \mathbb{R}^{n+1}$ is one of the reasons for the usefulness of Clifford algebras. Another reason is that complex analysis extends to higher dimensions as we shall now see.

3. Clifford analysis

In this section we wish to extend the results of complex analysis to Clifford algebras. Classically we would begin with C^1 functions

$$f : \Omega \to \mathbb{C} \quad \text{where } \Omega \text{ is an open subset of } \mathbb{C} \; ;$$

here, then, we consider C^1 functions

$$f : \Omega \to \mathbb{R}_{(n)} \quad \text{where } \Omega \text{ is an open subset of } \mathbb{R}^{n+1} \; .$$

We define

$$D = \sum_{j=0}^{n} \frac{\partial}{\partial x_j} e_j$$

acting on such f by

$$Df = \sum_{j=0}^{n} \sum_{S} \frac{\partial f_S}{\partial x_j} e_j e_S \quad \text{where} \quad f = \sum_S f_S e_S \ ;$$

by analogy with previous usage we define $\bar{D} = \frac{\partial}{\partial x_0} e_0 - \ldots - \frac{\partial}{\partial x_n} e_n$

Thus

$$\bar{D}D = D\bar{D} = \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right) e_0 = \Delta \ .$$

Corresponding to the notion of holomorphic we define f to be *monogenic* if $Df = 0$.

PROPOSITION 2. *If* f *is monogenic, then* f_S *is harmonic for all* S

Proof. As $\Delta f = \bar{D}Df = 0$, we have $\Delta f_S = 0$.

EXAMPLES.

(1) $n = 1$: $\mathbb{R}^{1+1} = \mathbb{R}_{(1)} = \mathbb{C}$.

Here

$$Df = \left(\frac{\partial}{\partial x_0} e_0 + \frac{\partial}{\partial x_1} e_1\right)(f_0 e_0 + f_1 e_1)$$

$$= \left(\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1}\right) e_0 + \left(\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0}\right) e_1$$

and so

f is monogenic iff $\frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_1}$ and $\frac{\partial f_0}{\partial x_1} = -\frac{\partial f_1}{\partial x_0}$;

i.e. iff $f_0 + if_1$ is holomorphic.

(2) $n = 3$: We consider the special functions $f = f_1 e_1 + f_2 e_2 + f_3 e_3$. Thus

$$Df = \left(\frac{\partial}{\partial x_0} e_0 + \ldots + \frac{\partial}{\partial x_3} e_3\right)(f_1 e_1 + f_2 e_2 + f_3 e_3)$$

$$= -\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) e_0 + \frac{\partial f_1}{\partial x_0} e_1 + \frac{\partial f_2}{\partial x_0} e_2 + \frac{\partial f_3}{\partial x_0} e_3$$

$$+ \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) e_1 e_2 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) e_2 e_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) e_1 e_3 \ .$$

Therefore

f is monogenic iff f is independent of x_0 , $\nabla \cdot f = 0$ and $\nabla \times f = 0$.

Clearly D is connected with the idea of differentiating k-forms, though we shall not go into details here.

<u>EXAMPLES OF MONOGENIC FUNCTIONS</u>. We first give what amounts to a non--example:

(0) $f(x) = x = x_0e_0 + x_1e_1 + \ldots + x_ne_n$.

Here $Df = e_0 - e_0 - \ldots - e_0 = (1-n)e_0 = 0$ iff $n = 1$. Thus the identity function is only monogenic on the complex numbers. In higher dimension its role is taken by the following functions.

(1) $f_j(x) = x_{(j)} = x_je_0 - x_0e_j$ is monogenic for $1 \leq j \leq n$.

Note that if $x_0 = 0$ then $f_j(x) = x_{(j)} = x_j$. From these, we build the following functions, also monogenic for all n :

(2) $f_{ik}(x) = \frac{1}{2}(x_{(j)}x_{(k)} + x_{(k)}x_{(j)})$.

(3) For each multi-index $\alpha = (\alpha_1 \ldots \alpha_n)$, where as usual the α_j's are non-negative integers and $|\alpha| = \alpha_1 + \ldots + \alpha_k$, we define

$$V_\alpha(x) = \frac{1}{|\alpha|!}\sum_\sigma \Pi\sigma(\underbrace{x_{(1)},\ldots,x_{(1)}}_{\alpha_1}, \underbrace{x_{(2)}\ldots x_{(2)}}_{\alpha_2},\ldots,\underbrace{x_{(n)}\ldots x_{(n)}}_{\alpha_n})$$

where the sum is over all permutations σ of $|\alpha|$ elements, and Π multiplies the elements of the resulting string together. Note that if $x = 0$ then $V_\alpha(x) = x^\alpha$.

(4) The functions $f(x) = \sum_\alpha c_\alpha V_\alpha(x)$ are monogenic on the domain of convergence, as are $f(x) = \sum_\alpha c_\alpha V_\alpha(x-a)$ for fixed c_α , a . Indeed, our intuition from complex analysis does not lead us astray. Every monogenic function is of this form in some nbhd of each $a \in \Omega$.

From the comments in examples (3) and (4), we find that every real analytic function g defined on an open set $\tilde{\Omega} \subset \mathbb{R}^n$ can be extended to a monogenic function f defined on an open set $\Omega \subset \mathbb{R}^{n+1}$, where $\Omega \cap \mathbb{R}^n = \tilde{\Omega}$.

(5) For $\Omega = \mathbb{R}^{n+1} \setminus \{0\}$ define

$$E(x) = \bar{x}|x|^{-(n+1)} = \begin{cases} \frac{1}{(1-n)}\bar{D}\frac{1}{|x|^{n-1}} , & n = 2,3,\ldots \\ \bar{D}(\log|x|) , & n = 1 . \end{cases}$$

Since $\frac{1}{|x|^{n-1}}$ and $\log|x|$ are harmonic E is monogenic.

(6) For $y \in \mathbb{R}^{n+1}$, define $E_y(x) = E(x-y) = \frac{\overline{x-y}}{|x-y|^{n+1}}$, $x \neq y$

Then E_y is monogenic.

(7) We use $E_y(x)$ as a generalization of the Cauchy kernel $(z-\zeta)^{-1}$. Let Σ be a smooth n-dimensional oriented submanifold of $\mathbb{R}^{n+1}$, let $n(y)$ be a consistent unit normal at $y \in \Sigma$, and let f be integrable on Σ. Define

$$(Tf)(x) = \frac{1}{\sigma_n} \int_\Sigma \frac{\overline{x-y}}{|x-y|^{n+1}} n(y) f(y)\, dS_y , \quad x \notin \Sigma ,$$

where σ_n is the area of the n-sphere in $\mathbb{R}^{n+1}$.

Monogenicity follows by differentiability under the integral sign. Note that

$$(Tb)(x) = \frac{1}{\sigma_n} \int_\Sigma \frac{\langle x-y, n(y)\rangle}{|x-y|^{n+1}} f(y)\, dS_y$$

$$+ \frac{1}{\sigma_n} \sum_{0 \leq j < k \leq n} e_k e_j \int_\Sigma \frac{(x-y)_j n_k - (x-y)_k n_j}{|x-y|^{n+1}} f(y)\, dS_y .$$

The first term is the harmonic function obtained by applying the double-layer potential operator to f. Indeed, by Proposition 2, each term is harmonic.

Cauchy's theorem can be generalized to higher dimensions. Suppose f is monogenic on Ω, and that Ω_0 is a bounded open subset of Ω with smooth boundary Σ. For $y \in \Sigma$, let $n(y)$ denote the inward pointing normal. Then

$$(Tf)(x) = \begin{cases} f(x), & x \in \Omega_0 \\ 0, & x \in \mathbb{R}^{n+1} \setminus \overline{\Omega}_0 . \end{cases}$$

4. The Cauchy singular integral operator

As well as the operator T defined above, we can define the principle value Cauchy operator $\mathbf{T}$ on an n-dimensional surface Σ in $\mathbb{R}^{n+1}$. For a smooth function $u : \Sigma \to \mathbb{R}$, we define

$$(\mathbf{T}u)(x) = \frac{2}{\sigma_n} \text{ p.v.} \int_\Sigma \frac{\overline{x-y}}{|x-y|^{n+1}} n(y) u(y)\, dS_y .$$

Again we see that $\mathbf{T}_0$, the scalar part of $\mathbf{T}$, is the double-layer potential operator on Σ.

Let us analyse the special case when $\Sigma = \mathbb{R}^n$ and $n(y) = -e_0$.

Then $\overline{(x-y)}n(y) = (x-y)e_0 = x - y$, so

$$(Tu)(x) = \frac{2}{\sigma_n} \text{ p.v.} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} u(y)\, dy = \sum_{k=1}^{n} e_k (R_k u)(x) ,$$

where R_k are the Riesz potential operators.

To consider the L_2 theory of these operators, we let H denote the Hilbert space,

$$H = L_2(\mathbb{R}^n)_{(n)} = \{u = \sum_S u_S e_S \mid u_S \in L_2(\mathbb{R}^n)\}$$

with inner product $(u,v) = \sum_S (u_S, v_S)$. Let $\underset{\sim}{D}$ denote the Dirac operator,

$$\underset{\sim}{D} = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$$

with domain the Sobolev space $H^1(\mathbb{R}^n)_{(n)}$.

It is not hard to verify that $\underset{\sim}{D}$ is a self-adjoint operator with spectrum $\sigma(\underset{\sim}{D}) = \mathbb{R}$. So $f(\underset{\sim}{D}) = L(H)$ for all bounded Borel functions $f : \mathbb{R} \to \mathbb{C}$, and $||f(\underset{\sim}{D})|| = ||f||_\infty$. In particular

$$\text{sgn } (\underset{\sim}{D}) = \frac{\underset{\sim}{D}}{|\underset{\sim}{D}|} = \frac{\sum D_k e_k}{(R^2)^{1/2}} = \sum_{k=1}^{n} \frac{D_k}{(-\Delta)^{1/2}} e_k = \sum_{k=1}^{n} R_k e_k = T .$$

That is, the Cauchy operator T is precisely $\text{sgn } (\underset{\sim}{D})$. When $n = 1$, this is well known, for T is the Hilbert transform. But it is somewhat surprising that the Riesz transform can be represented as the signum of a self-adjoint operator.

Using the functional calculus for self-adjoint operators, we can also write T as, for example,

$$T = \text{sgn } (\underset{\sim}{D}) = \frac{16}{\pi} \int_0^\infty \Psi^3(t\, \underset{\sim}{D}) \frac{dt}{t} ,$$

where the integral is defined using the strong operator topology, and

$$\Psi(\lambda) = \lambda(1 + \lambda^2)^{-1} .$$

This is because

$$\int_0^\infty \Psi^3(t\lambda) \frac{dt}{t} = \frac{\pi}{16} \text{ sgn } (\lambda) .$$

for real numbers λ .

5. Lipschitz surfaces

We leave now the case when $\Sigma = \mathbb{R}^n$, and suppose that Σ is the graph of a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$. That is, $\Sigma = \{g(\underset{\sim}{x})e_0 + \underset{\sim}{x} \mid \underset{\sim}{x} \in \mathbb{R}^n\}$, and T is defined as above for functions $u : \Sigma \to \mathbb{C}$.

THEOREM. T *is* L_2*-bounded.*

COROLLARY. *The double-layer potential operator* T_0 *is* L_2*-bounded.*

In the case when $n = 1$ this theorem and its corollary was first proved in the paper [2] of Coifman, McIntosh and Meyer. It was also shown that the higher dimensional result could be reduced to the one--dimensional estimates of [2] using the Calderón rotation method. This result was first used in potential theory by Verchota. Subsequently Coifman discovered the significance of the operators T and $\underset{\sim}{D}$ defined above, and asked whether the one-dimensional proof could be generalized to give a direct proof of the theorem in higher dimensions. This was shown to be the case for surfaces with small Lipschitz constant by Murray [4], and then for all Lipschitz graphs by the author.

To give some idea of the proof, let $b = \underset{\sim}{D}g$, and let $A = (I - B)^{-1}\underset{\sim}{D}$ where B is the multiplication operator on H defined by $Bu = bu$. Then the spectrum of A is contained in a double sector S_ω for some $\omega < \pi/2$, where $S_\omega = \{z \in \mathbb{C} \mid \arg(z) \leq \omega$ or $\arg(-z) \leq \omega\}$. As in the case when $\Sigma = \mathbb{R}$ we find that

$$T = \frac{16}{\pi}\int_0^\infty Q_t^3 \frac{dt}{t},$$

where $Q_t = tA(I + t^2A^2)^{-1}$. Although A is not self-adjoint, we can still think of T as $\operatorname{sgn}(A)$ for the signum function defined to be $+1$ on the right sector of S_ω and -1 on the left sector. Also, $\|Q_t\| \leq \kappa < \infty$ for all t.

So, for $u, v \in H$,

$$|(Tu,v)| = \left|\int_0^\infty (Q_tQ_tu,\ Q_t^*v)\frac{dt}{t}\right|$$

$$\leq \kappa\left\{\int_0^\infty \|Q_tu\|^2\frac{dt}{t}\right\}^{1/2}\left\{\int_0^\infty \|Q_t^*v\|^2\frac{dt}{t}\right\}^{1/2}.$$

Hence the boundedness of T will be a consequence of the square function estimate

$$\{\int_0^\infty ||Q_t u||^2 \frac{dt}{t}\}^{1/2} \leq c||u|| ,$$

together with a dual estimate. If $||B|| < 1$, then

$$Q_t = \frac{i}{2}(R_t - R_{-t})$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\{(\underset{\sim}{R}_t B)^{k-s}\,\underset{\sim}{Q}_t(B\underset{\sim}{P}_t)^s + (\underset{\sim}{R}_{-t}B)^{k-s}\underset{\sim}{Q}_t(B\underset{\sim}{P}_t)^s\}(I - B) ,$$

where $R_t = (I + itA)^{-1}$, $\underset{\sim}{R}_t = (I + it\underset{\sim}{D})^{-1}$, $\underset{\sim}{P}_t = (I + t^2\underset{\sim}{D}^2)^{-1}$, and $\underset{\sim}{Q}_t = t\underset{\sim}{D}(I + t^2\underset{\sim}{D}^2)^{-1}$. So the square function estimate for Q_t is a consequence of the following estimates,

$$\{\int_0^\infty ||\underset{\sim}{Q}_t(B\underset{\sim}{P}_t)^k u||^2 \frac{dt}{t}\}^{1/2} \leq c(1 + k)||B||^k ||u||_2 ,$$

which are similar to those proved in [2] when $n = 1$. When $||B|| > 1$, a somewhat different expansion is needed. Details will appear in [3].

We conclude with the remark that Clifford algebras have allowed us to replace n-tuples of operators by single operators and hence use spectral theory and the functional calculus in this setting.

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