Hans P. Heinig

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Weighted Inequalities in Fourier Analysis

H. P. Heinig

Hamilton, Ontario

Introduction. Inequalities involving the Fourier transform and its variants have been an essential part of Fourier analysis from its early beginnings. This is not surprising because the size and estimate of the Fourier transform in various function spaces is significant in the development of harmonic analysis and is underscored by the numerous applications they yield in science and engineering. The first L^p–estimate of the Fourier transform involving power weights are the results of Paley–Titchmarsh (1934) with extensions by Pitt (1937), Stein (1956) and Rooney (1966). Somewhat more general weights were considered by Hirschman (1957) and Flett (1973). Only in 1978, Muckenhoupt ([22]) formally posed the problem of characterizing for given indices p and q those non–negative weight functions u and v for which the inequality

$$\|\hat{f}\|_{q,u} \le C\|f\|_{p,v}$$

holds, for all $f \epsilon L^1$. This problem has been studied by a number of workers, including Muckenhoupt, and, although the complete solution is still illusive, much progress has been made.

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Many results regarding the weighted Fourier transform inequality follow from interpolation arguments and those in turn depend on results about the Hardy operator, its dual as well as variants and generalizations. It is therefore also natural to study in this connection weighted estimates for the Hardy operator. In 1986 a survey article on these subjects appeared in the Proceedings of the Spring School held in Litomyšl, Czechoslovakia (Nonlinear Analysis, Function Spaces and Applications v.3; M. Krbec, A. Kufner, J. Rákosník, ed., Teubner Texte zur Math. Bd. 43, Leipzig 1986). The article which follows, is an attempt to describe the progress which has been made in this area since the Litomyšl paper was written.

This work is divided into three parts. The first describes recent generalizations and extensions of Hardy's inequality. We describe how weighted gradient inequalities follow in a natural way from Hardy's inequality. In a limiting case of the weighted L^p inequality for the Hardy averaging operator, we obtain weighted characterizations of exponential inequalities of this operator. These results have higher dimensional analogues which together with their discrete versions seem to be new. Then we describe results on weighted Hardy inequalities for decreasing functions. These characterizations lead to weight characterizations of more general operators, such as the Laplace transform and the Riemann–Liouville operator, on classical Lorentz spaces. Section 2 contains various weighted and measure weighted Fourier inequalities. In particular, we point out that certain weighted Fourier inequalities are equivalent to weighted Hardy inequalities. The last section describes some application of these results to Fourier restriction and extensions theorems as well as uncertainty inequalities.

It is a pleasure to thank the organizers, Professors M. Krbec, A. Kufner and J. Rákosník for the invitation to participate in this conference, for their fine organization and generous hospitality which made this visit to Czechoslovakia especially enjoyable.

Before we begin our discussion, we collect some notations and definitions used in the sequel.

As usual we write for $(-\infty, \infty) = \mathbb{R}$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}^n = \{x = (x_1,\ldots,x_n),\ x_i \in \mathbb{R}, i = 1,2,\ldots,n\}$ and $dx = dx_1dx_2\ldots dx_n$. The Fourier transform of f on $\mathbb{R}^n$ is defined by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-ix\cdot y} f(y)dy, \quad x \in \mathbb{R}^n,$$

whenever the integral converges. L^p denote the usual Lebesgue spaces of functions with

$$\|f\|_p^p \equiv \int_{\mathbb{R}^n} |f(x)|^p dx < \infty, \quad 0 < p < \infty.$$

If u is a non-negative weight function, then we write $f \in L^p_u$, if $\|f\|_{p,u} = \|fu^{1/p}\|_p < \infty$. Similarly, if μ is a measure, then $f \in L^p_\mu$, if $\|f\|_{p,\mu} < \infty$. The conjugate index p' of p, $0 < p < \infty$, is defined by $\frac{1}{p} + \frac{1}{p'} = 1$ with $p' = \infty$ if $p = 1$.

$C_c(\mathbb{R})$ denotes the space of continuous functions of compact support on $\mathbb{R}$, while $C_0(\mathbb{R})$ is the space of continuous functions $f(x)$ vanishing as $|x| \longrightarrow \infty$. A measure μ on $\mathbb{R}$ is a linear functional on $C_c(\mathbb{R})$. μ is positive if $<\mu, f> \geq 0$ for all $f \in C_c(\mathbb{R})$, $\mathscr{S}(\mathbb{R})$, respectively, $\mathscr{S}(\mathbb{R}^n)$ are the spaces of Schwartz functions on $\mathbb{R}$, respectively $\mathbb{R}^n$.

We write $f\downarrow$ if f is decreasing on $\mathbb{R}^+$ and decreasing means non-increasing.

$B_r(x_0) \subset \mathbb{R}^n$ is the n-ball centered at x_0 with radius $r > 0$. $\Sigma_{n-1}(\rho) = \{x \in \mathbb{R}^n: |x| = \rho\}$ is the n-sphere of radius ρ, while $\Sigma = \Sigma_{n-1}(1)$. $d\sigma$ or $d\tau$ are the surface measures on these spheres, and χ_E is the characteristic function of the set E. We shall write $G \approx H$, if there are positive constants C_0, C_1 such that $C_0 \leq G/H \leq C_1$. Constants will be denoted by C,B and A (at times with subscripts), which may be different from place to place. Finally, inequalities $\|Tf\| \leq C\|f\|$ are interpreted to mean that if the right side is finite, so is the left, and the inequality holds.

1. Recent generalizations and variants of Hardy's inequality. The classical Hardy inequality states that if P is the averaging operator

$$(Pf)(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0, \tag{1.1}$$

then P is bounded on $L^p(\mathbb{R}^+)$, $p > 1$. This results and its generalizations and variants find numerous applications in many branches of mathematics. For example in the theory of differential equations, approximation theory, interpolation theory, the study of function spaces, and Fourier analysis to mention a few. During the last twenty years, Hardy's inequality was extended to general measures and in particular to measures generated by weight functions. The general result in terms of weights may be summarized in Theorem 1.1, which will be the starting point of our discussion.

Theorem 1.1. Suppose $0 < p, q \le \infty$, $p \ge 1$. If f is non-negative and u,v are weights on $(0, \infty)$, then the inequality

$$\left[\int_0^\infty u(x)\left(\int_0^x f(t)dt\right)^q dx\right]^{1/q} \le C\left[\int_0^\infty v(x)\, f(x)^p dx\right]^{1/p}$$

holds, if and only if

(i) for $1 \le p \le q \le \infty$, $\sup_{r>0} \left(\int_r^\infty u(x)dx\right)^{1/q} \left(\int_0^r v(x)^{1-p'}dx\right)^{1/p'} < \infty$;

(ii) for $0 < q < p < \infty$, $p \ge 1$, $\left\{\int_0^\infty\left[\left(\int_x^\infty u(t)dt\right)^{1/q}\left(\int_0^x v(t)^{1-p'}dt\right)^{1/q'}\right]^r v(x)^{1-p'}dx\right\}^{1/r} < \infty$,
where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

There is of course also an obvious dual result which is obtained easily via changes of variables. For a proof of this result together with an excellent historical survey and many applications we refer to the forthcoming book by Kufner and Opic [18].

It may be instructive to illustrate how Theorem 1.1 can be applied to a gradient type differential inequality in $\mathbb{R}^n$. Note that the averaging operator P of (1.1) and its dual Q can be written in the form

$$(Pf)(x) = \int_0^1 f(xt)dt \qquad (Qf)(x) = \int_1^\infty f(xt) \frac{dt}{t}$$

But in this form the operator is also well defined for $x \in \mathbb{R}^n$. Now if Q^* is defined by $(Q^*f)(x) = -x \cdot (\nabla f)(x)$, $x \in \mathbb{R}^n$ then for any $f \in C_0^1(\mathbb{R}^n)$ we get

$$Q(Q^*f)(x) = -\int_1^\infty (tx)\cdot(\nabla f)(tx) \frac{dt}{t} = -\int_1^\infty \frac{d}{dt} f(tx)dt = f(x).$$

Now using polar coordinates it follows from this, that

$$\int_{\mathbb{R}^n} |f(x)|^q u(x)dx = \int_{\mathbb{R}^n} u(x)\Big|\int_1^\infty (Q^*f)(xt) \frac{dt}{t}\Big|^q dx = \int_\Sigma\int_0^\infty u(s\sigma)s^{n-1}\Big|\int_1^\infty (Q^*f)(\sigma st) \frac{dt}{t}\Big|^q ds d\sigma$$

$$= \int_\Sigma\int_0^\infty u(s\sigma)s^{n-1}\Big|\int_s^\infty (Q^*f)(\sigma y) \frac{dy}{y}\Big|^q ds d\sigma.$$

Since the inner integral corresponds to the one dimensional dual operator one obtains on applying the dual of Theorem 1.1 (i) the following gradient inequality of Sinnamon [28]:

<u>Theorem 1.2.</u> Suppose $1 < p < \infty$, $0 < q < \infty$, then for all $f \in C_0^\infty(\mathbb{R}^n)$

$$\Big(\int_{\mathbb{R}^n} |f(x)|^q v(x)dx\Big)^{1/q} \le C\Big(\int_{\mathbb{R}^n} |x\cdot(\nabla f)(x)|^p u(x)dx\Big)^{1/p}$$

holds, if and only if

(i) for $p = q$, $\sup_{x \in \mathbb{R}^n} \left(\int_0^1 v(tx) t^{n-1} dt \right)^{1/p} \left(\int_1^\infty [u(tx)t^n]^{1-p'} \frac{dt}{t} \right)^{1/p'} < \infty$

(ii) for $0 < q < p$, $p > 1$

$$\left\{ \int_{\mathbb{R}^n} \left[\left(\int_0^1 v(tx) t^{n-1} dt \right)^{1/q} \left(\int_1^\infty [u(tx)t^n]^{1-p'} \frac{dt}{t} \right)^{1/q'} \right]^r u(x)^{1-p'} dx \right\}^{1/r} < \infty$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$

It should be remarked here that in the case $1 < p < q < \infty$, not only does the proof of this result fail but also the inequality (c.f. [28, Theorem 3.4]).

If the indices $p, q \in (0, 1)$ then either the domain spaces of the Hardy operator must be replaced by the boundary values of functions in weighted Hardy spaces in order to get weighted inequalities ([12]) or the inequalities are in the opposite direction ([3]). For example in case $q \leq p < 0$ the following holds:

<u>Theorem 1.3.</u> ([3, Theorem 1]). Let $q \leq p < 0$, u,v and f positive a.e. and suppose

$$K(r) \equiv \left(\int_0^r u(x) dx \right)^{1/q} \left(\int_0^r v(x)^{1-p'} dx \right)^{1/p'}$$

Then

$$\left(\int_0^\infty v(x) f(x)^p dx \right)^{1/p} \leq C \left(\int_0^\infty u(x) \left(\int_0^\infty f(t) dt \right)^q dx \right)^{1/q} \qquad (1.2)$$

provided $K(r)$ is non-decreasing and $\inf_{r>0} K(r) \equiv B > 0$. Conversely, (1.2) implies $\inf K(r) \equiv B > 0$.

1.1. Inequalities related to those of Hardy's. We now discuss weighted inequalities which may be considered limiting cases of Hardy's inequality. Indeed we characterize the weights for which certain exponential inequalities hold. Unlike Hardy's inequality the higher dimensional generalizations of these results follow in a straight–forward manner. We give in addition a discrete form which yields a weighted characterization of Carleman's inequality. The results of this section have been obtained jointly with R. Kerman and M. Krbec.

Theorem 1.1 with $p = q$ shows (with explicit constant) that

$$\int_0^\infty u(x)\left(x^{-1}\int_0^x f(t)^{1/p}\,dt\right)^p dx \le B_p^p\, p(p')^{p-1}\int_0^\infty v(x)f(x)dx \qquad (1.3)$$

holds, if and only if

$$B_p^p \equiv \sup_{r>0}\left(\int_r^\infty x^{-p}u(x)dx\right)\left(\int_0^r v(x)^{1-p'}dx\right)^{1/p'} < \infty.$$

Now if $p \longrightarrow \infty$, then an application of Fatou's lemma shows that (1.3) implies

$$\int_0^\infty u(x)\exp\left(x^{-1}\int_0^x \ln f(t)dt\right)dx \le C\int_0^\infty v(x)f(x)dx, \qquad (1.4)$$

provided $\lim B_p^p p(p')^{p-1} = C$ as $p \longrightarrow \infty$. In this sense (1.4) might be considered the limiting case of Hardy's inequality. Of course there is no reason to assume that $B_p^p p(p')^{p-1} \longrightarrow C$ as $p \longrightarrow \infty$.

Besides the operator P of (1.1) we define Q_α, $\alpha > 0$, as follows:

$$(Q_\alpha f)(x) = \alpha x^\alpha \int_x^\infty t^{-\alpha-1}f(t)dt.$$

Theorem 1.4. Let u, v and f be positive a.e. and

$$w(x) = u(x)\exp[(P \ln \tfrac{1}{v})(x)], \quad x > 0.$$

Then, the following are equivalent:

(i) $\int_0^\infty u(x)\exp(P\ \ln f)(x)dx \le C_1 \int_0^\infty v(x)f(x)dx$

(ii) $(Pw)(x) + \alpha^{-1}(Q_\alpha w)(x) \le C_2, \quad \alpha > 0$

(iii) $(Pw) \in L^\infty$

(iv) $\int_0^\infty w(x)(Pf^{1/p})(x)dx \le C_3 \int_0^\infty f(x)dx, \quad (p > 2)$

where C_3 is independent of p.

Proof. (i) ⇒ (ii). Write $f = \frac{g}{v}$, where $g(x) = t^{-1}\chi_{(0,t)}(x) + x^{-\alpha-1}e^{-1-\alpha}t^\alpha\chi_{(t,\infty)}(x)$, $t > 0$, $\alpha > 0$. Substituting into (i) the right side becomes

$$C_1\left[t^{-1}\int_0^t dx + e^{-1-\alpha}t^\alpha\int_t^\infty x^{-\alpha-1}dx\right] = C_1[1 + e^{-1-\alpha}/\alpha],$$

while the left side takes the form

$$\int_0^\infty u(x)\exp(P\ \ln \tfrac{g}{v})(x)dx = \int_0^\infty u(x)\exp(P\ \ln \tfrac{1}{v})(x)\exp(P\ \ln g)dx$$

$$= \int_0^\infty w(x)\exp\left(x^{-1}\int_0^x \ln g(y)dy\right)dx = \left(\int_0^t + \int_t^\infty\right) \equiv I_1 + I_2$$

respectively. With the above defined g

$$I_1 = \int_0^t w(x)\exp\left[\frac{1}{x}\int_0^x \ln\left(\frac{1}{t}\right)dy\right]dx = \frac{1}{t}\int_0^t w(x)dx = (Pw)(t)$$

and

$$I_2 = \int_t^\infty w(x)\exp\left(x^{-1}\int_0^t \ln\left(\frac{1}{t}\right)dy\right)\exp\left(x^{-1}\int_t^x \ln\left(e^{-1-\alpha}t^\alpha y^{-\alpha-1}\right)dy\right)dx$$

$$= \int_t^\infty w(x)\exp\left[x^{-1}t\ \ln\left(\frac{1}{t}\right) + \left(1-\frac{t}{x}\right)\ln\left(e^{-1-\alpha}t^\alpha\right) - (\alpha+1)x^{-1}\int_t^x \ln y\,dy\right]dx$$

$$= \int_t^\infty w(x)\exp\left[x^{-1}t\ \ln\left(\frac{1}{t}\right) + \left(1-\frac{t}{x}\right)\left(-1-\alpha\right) + \alpha\left(1-\frac{t}{x}\right)\ln t - x^{-1}(\alpha+1)(x\ln x - x + t - t\ln t)\right]dx$$

$$= \int_t^\infty w(x)\exp\left[x^{-1}t\ \ln\left(\frac{1}{t}\right) + \left(1-\frac{t}{x}\right)\ln t - \left(\alpha+1\right)\ln x + x^{-1}\ t(\alpha+1)\ln t\right]dx$$

$$= \int_t^\infty w(x)\exp\left[\alpha\ln t - (\alpha+1)\ln x\right]dx = t^\alpha\int_t^x x^{-\alpha-1}\,w(x)dx$$

$$= \alpha^{-1}(Q_\alpha w)(t).$$

Therefore, $(Pw)(t) + \alpha^{-1}(Q_\alpha w)(t) \le C_1[1 + e^{-1-\alpha}/\alpha] \equiv C_2$ and (ii) holds.

(ii) ⇒ (iii) is obvious. To prove (iii) ⇒ (iv) note that by Hölder's inequality with $q > 1$

$$(Ph)(x) \le [(Ph^q)(x)]^{1/q} \le \|h\|_q/x^{1/q},$$

so that

$$\int_{\{x:(Ph)(x)>\lambda\}} w(x)dx \le \int_{\{x:(\|h\|_q/\lambda)^q>x\}} w(x)dx = \int_0^{(\|h\|_q/\lambda)^q} w(x)dx \le C(\|h\|_q/\lambda)^q .$$

Here the last inequality is implied by (iii) with C independent of q. Hence the operator P is bounded from L^q to weighted "weak" L^q_w, $q > 1$ with norm independent of q. Now let $p > 2$, then we apply the Marcinkiewicz interpolation theorem with $q_0 = p/2$ and $q_1 = 3p/2$, $(\theta = 3/4)$ to obtain

$$\int_0^\infty w(x)\ (Ph)(x)^p dx \le C_3 \int_0^\infty h(x)^p dx$$

where C_3 is independent of p. With $h^p = f$ this implies (iv).

Finally (iv) $\Rightarrow$ (i) follows from Fatou's lemma and the fact that

$$\lim_{p\to\infty} (Pf^{1/p})^p(x) = \exp(P \ln f)(x) \qquad ([33; \text{p. } 344, 5c]).$$

Remark 1.5. If (iii) holds, then $(Pw)(x) \le C$, so that $(Q_\alpha(Pw))(x) \le (Q_\alpha C)(x) = C$. Thus

$$C \ge (Q_\alpha(Pw))(x) = \alpha x^\alpha \int_x^\infty y^{-\alpha-2}\left(\int_0^y w(t)dt\right)dy$$

$$= \alpha x^\alpha \int_x^\infty y^{-\alpha-2}\left[\int_0^x w(t)dt + \int_x^y w(t)dt\right]dy = \frac{\alpha}{\alpha+1}(Pw)(x) + \alpha x^\alpha \int_x^\infty w(t) \int_t^\infty y^{-\alpha-2}dydt$$

$$= \frac{\alpha}{\alpha+1}(Pw)(x) + \frac{1}{\alpha+1}(Q_\alpha w)(x).$$

Therefore, $Pw \in L^\infty$, if and only if $Q_\alpha w \in L^\infty$ and we obtain

<u>Corollary 1.6.</u> If u, v and f are positive and

$$w(x) = u(x)\exp(P\ln 1/v)(x), \quad x > 0,$$

then

$$\int_0^\infty u(x)\exp\left(x^{-1}\int_0^x f(y)dy\right)dx \le C\int_0^\infty v(x)\exp(f(x))dx,$$

if and only if for any $\alpha > 0$, $\sup_{x>0} \alpha^{-1}(Q_\alpha w)(x) < \infty$.

A simple calculation shows that $u(x) = v(x) = x^\beta$, β real, satisfy the weight condition and so do the functions

$$u(x) = e^{\beta x/2},\ v(x) = e^{\beta x},\ \beta \text{ real.}$$

The characterization of the weights for the exponential inequality for the averaging operator in higher dimensions carries over in a straight forward way unlike the corresponding characterizations in L^p–spaces, where the n–dimensional result, $n > 2$, is still open. Here we simply state the two dimensional case of the exponential inequality.

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and P^2, Q_α^2 be defined by

$$(P^2f)(x) = x_1^{-1}x_2^{-1}\int_0^{x_1}\int_0^{x_2} f(y)dy,\ (Q_\alpha^2 f)(x) = \alpha_1\alpha_2 x_1^{\alpha_1}x_2^{\alpha_2}\int_{x_1}^\infty\int_{x_2}^\infty y_1^{-\alpha_1-1}y_2^{-\alpha_2-1}f(y)dy$$

$\alpha_1 > 0$, $\alpha_2 > 0$, then the following holds:

Theorem 1.7. Let u, v and f be positive a.e. on $\mathbb{R}^2_+$, then

$$\int_{\mathbb{R}^2_+} u(x) \exp(P^2 \ln f)(x)dx \le C \int_{\mathbb{R}^2_+} v(x) f(x) \, dx$$

if and only if $\sup_{x_1>0, x_2>0} (Q^2 w)(x) < \infty$, where $w(x) = u(x) \exp(P^2 \ln 1/v)(x)$.

There are further extensions of these results to more general operators with positive kernels satisfying certain homogeneity conditions. The Laplace transform is a special case of these. We shall not discuss these generalizations here, instead we consider the discrete case and give a variant of the arithmetic-geometric mean inequality.

Recall that if $a_1, a_2, \ldots$; are positive real numbers, then Carleman's inequality asserts that

$$\sum_{n=1}^{\infty} [a_1 a_2 \ldots a_n]^{1/n} \le e \sum_{n=1}^{\infty} a_n, \tag{1.5}$$

where the constant e is sharp. There are a number of weighted generalizations of this inequality, some with sharp constants (c.f. [10], [13], [19]). The next result characterizes weights for which a weighted inequality corresponding to (1.5) holds.

Theorem 1.8. Let $\{u_n\}, \{v_n\}, \{a_n\}$ be sequences of positive numbers and suppose

$$w_n = u_n \exp\left[\frac{1}{n} \sum_{k=1}^{n} \ln(1/v_k)\right] \quad n = 1,2,\ldots ;$$

Then

$$\sum_{n=1}^{\infty} u_n [a_1 a_2 \ldots a_n]^{1/n} \le C \sum_{n=1}^{\infty} v_n a_n \tag{1.6}$$

if and only if

$$\sup_{k\geq 1} k^{\alpha} \sum_{n=k}^{\infty} n^{-\alpha-1} w_n \equiv A < \infty,\ \alpha > 0. \tag{1.7}$$

Moreover

$$\alpha A/(\alpha + e^{-1-\alpha}) \leq C \leq e^{\alpha} A.$$

Proof. Let $b_n = a_n v_n$, $n = 1,2,\ldots$; then (1.6) is equivalent to

$$\sum_{n=1}^{\infty} w_n \exp\left[\frac{1}{n} \sum_{k=1}^{n} \ln b_k\right] \leq C \sum_{n=1}^{\infty} b_n. \tag{1.8}$$

Let $f(t) = b_k$, if $k - 1 < t \leq k$, $k = 1,2,\ldots$; and zero otherwise. Since

$$\int_0^1 \ln(y^{\alpha})dy = -\alpha$$

the left side of (1.8) becomes

$$\sum_{n=1}^{\infty} w_n \exp\left[\frac{1}{n} \sum_{n=1}^{n} \int_{k-1}^{k} \ln f(t)dt\right] = \sum_{n=1}^{\infty} w_n \exp\left[\frac{1}{n} \int_0^n \ln f(t)dt\right]$$

$$= \sum_{n=1}^{\infty} w_n \exp\left[\int_0^1 \ln f(ny)dy\right] = e^{\alpha} \sum_{n=1}^{\infty} w_n \exp\left[\int_0^1 \ln(y^{\alpha} f(ny))dy\right]$$

$$\leq e^{\alpha} \sum_{n=1}^{\infty} w_n \int_0^1 y^{\alpha} f(ny)dy = e^{\alpha} \sum_{n=1}^{\infty} w_n n^{-1-\alpha} \int_0^n x^{\alpha} f(x)dx$$

$$= e^{\alpha} \sum_{n=1}^{\infty} w_n n^{-1-\alpha} \sum_{k=1}^{n} \int_{k-1}^{k} x^{\alpha} f(x)dx = e^{\alpha} \sum_{n=1}^{\infty} w_n n^{-1-\alpha} \sum_{k=1}^{n} k^{\alpha} b_k$$

$$= e^{\alpha} \sum_{k=1}^{\infty} b_k \Big[k^{\alpha} \sum_{n=k}^{\infty} n^{-1-\alpha} w_n \Big] \le A e^{\alpha} \sum_{k=1}^{\infty} b_k .$$

Here the first inequality follows from Jensen's inequality. Hence (1.8) follows.

To show that (1.8) implies (1.7) let

$$b_k = \begin{cases} 1/k, & \text{if } n \le k \\ e^{-1-\alpha} k^{\alpha} n^{-1-\alpha}, & \text{if } n > k ; \end{cases}$$

$n = 1,2,\dots$; k fixed. Then

$$\sum_{n=1}^{\infty} b_n = \frac{1}{k} \sum_{n=1}^{k} 1 + e^{-1-\alpha} k^{\alpha} \sum_{n=k+1}^{\infty} n^{-\alpha-1} \le 1 + e^{-1-\alpha}/\alpha$$

and by (1.8)

$$C(1 + e^{-1-\alpha}/\alpha) \ge \sum_{n=1}^{k} w_n \exp\Big[\frac{1}{n} \sum_{j=1}^{n} \ln b_j\Big] + \sum_{n=k+1}^{\infty} w_n \exp\Big[\frac{1}{n} \sum_{j=1}^{n} \ln b_j\Big] \equiv I_1 + I_2,$$

respectively. But $b_j = 1/k$ if $j \le k$, so $I_1 = \frac{1}{k} \sum_{n=1}^{k} w_n$ and

$$I_2 = \sum_{n=k+1}^{\infty} w_n \exp\Big[\frac{1}{n} \Big(\sum_{j=1}^{k} \ln 1/k + \sum_{j=k+1}^{n} \Big(e^{-1-\alpha} k^{\alpha} / j^{\alpha+1} \Big)\Big)\Big]$$

$$= \sum_{n=k+1}^{\infty} w_n \exp(n^{-1} k \ln 1/k) \exp[n^{-1}(n-k-1) \ln(e^{-1-\alpha} k^{\alpha})] \cdot \exp\Big[-n(\alpha + 1) \sum_{j=k+1}^{n} \log j\Big].$$

But

$$\sum_{j=k+1}^{n} \log j \le -n + k - 1 + (n + 1)\ln n - (k + 1)\ln k$$

so that

$$I_2 \ge \sum_{n=k+1}^{\infty} w_n \exp[\ln(k^{\alpha}) + \ln(n^{-\alpha-1}) + n^{-1} \ln(k/n^{\alpha+1})]$$

$$\ge k^{\alpha} e^{-1-\alpha} \sum_{n=k+1}^{\infty} w_n n^{-\alpha-1}$$

since $n^{-1} \ln(k/n^{\alpha+1}) \ge -n^{-1}(\alpha + 1) \ln n \ge -\alpha-1$. It follows then that

$$C[1 + e^{-1-\alpha}/\alpha] \ge k^{-1} \sum_{n=1}^{k} w_n + e^{-\alpha-1} k^{\alpha} \sum_{n=k}^{\infty} w_n n^{-\alpha-1} - e^{-\alpha-1} w_k k^{-1}$$

$$\ge e^{-\alpha-1} k^{\alpha} \sum_{n=k}^{\infty} w_n n^{-\alpha-1}.$$

Hence $C \ge \dfrac{\alpha}{\alpha + e^{-1-\alpha}} A$, which proves the theorem.

Extensions of this results to higher dimensions follow as in the continuous case. In addition characterizations of weights in the directions of the results in [10], [13] and [19] are possible.

We now consider briefly more general operators than the averaging operator and their boundedness in weighted L^p spaces. Specifically, we consider the Riemann–Liouville fractional integral operator defined by

$$(P_{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t)dt, \qquad 0 < \alpha < \infty. \tag{1.9}$$

Recently Martin–Reyes and Sawyer [20] and independently Stepanov [31] proved that $P_\alpha: L^p_v \to L^q_u$, $1 \le p \le q < \infty$, $\alpha \ge 1$ is bounded, if and only if

$$\sup_{r>0} \left(\int_r^\infty u(t)(t/r)^{q(\alpha-1)}dt\right)^{1/q} \left(\int_0^r (r-t)^{p'(\alpha-1)}\, v(t)^{1-p'}dt\right)^{1/p'} \equiv A < \infty$$

and

$$\sup_{r>0} \left(\int_r^\infty u(t)(t-r)^{q(\alpha-1)}dt\right)^{1/q} \left(\int_0^r v(t)^{1-p'}dt\right)^{1/p'} \equiv B < \infty.$$

It should be noted that Stepanov's (equivalent) conditions are somewhat different, namely

$$\max_{\gamma=0,1} \sup_{r>0} \left(\int_r^\infty (t-x)^{(\alpha-1)(1-\gamma)}\, u(t)dt\right)^{1/q} \left(\int_0^r (r-t)^{(\alpha-1)\gamma p'} v(t)dt\right)^{1/p'} < \infty.$$

Stepanov further characterized the weights u, v for which $P_\alpha: L^p_v \to L^q_u$ is bounded in the index range $0 < q < p < \infty$, $p > 1$.

The corresponding characterizations of the weights for P_α when $0 < \alpha < 1$ has not yet been solved, however, sufficient conditions, similar to those given above are known in this case ([1], [14]). We shall return to this question shortly.

For far more general integral operators with positive kernels, weight characterization of a different kind were given in [16] (see also the literature cited there).

1.2. Weighted inequalities for operators of monotone functions.

If $1 < p = q < \infty$ and $u(t) = v(t)/t^p$, where $v(t)$ is defined by

$$v(t) = \begin{cases} 0 & \text{if } 1 < t < 2 \\ t^{-1/2} & \text{if } 0 < t \le 1 \text{ or } t \ge 2 \end{cases},$$

then clearly (i) of Theorem 1.1. fails. However, if f is a decreasing function, then as shown in [2]

$$\int_0^\infty v(x)\,(Pf)(x)^p dx \leq C \int_0^\infty v(x)\, f(x)^p dx, \quad f \geq 0.$$

It is therefore natural to consider the analogue of Theorem 1.1 for decreasing functions. The first characterization of weights in this direction is due to Ariño and Muckenhoupt. Their result is

Theorem 1.9. ([2; Theorem 1.7]). If $1 \leq p < \infty$ and $u(x) \geq 0$ then $\|Pf\|_{p,u} \leq C\|f\|_{p,u}$ holds for all non-negative decreasing f, if and only if for all $r > 0$

$$\int_r^\infty x^{-p} u(x)dx \leq C\, r^{-p} \int_0^r u(x)dx. \tag{1.10}$$

Let $\Lambda_p(u)$ denote the classical Lorentz spaces, that is, the set of measurable functions g on $\mathbb{R}^n$, such that

$$\|g\|_{\Lambda_p(u)} \equiv \left\{\int_0^\infty g^*(x)^p u(x)dx\right\}^{1/p} < \infty, \ 1 \leq p < \infty,$$

where g^* is the rearrangement of $|g|$, namely

$$g^*(x) = \inf\{s > 0;\ \mu(\{t\colon |g(t)| > s\}) \leq x\},$$

here μ denotes Lebesgue measure. It follows then from Theorem 1.9 that the Hardy–Littlewood maximal operator is bounded on $\Lambda_p(u)$ if and only if u satisfies (1.10).

There are several generalizations of Theorem 1.9 due to Braverman ([8]), Neugebauer ([23]), Sawyer ([27]) and Stepanov ([32]). Braverman defined the operator T_φ by

$$(T_\varphi f)(x) = x^{-1} \int_0^x f(t)\ \varphi(t/x)dt \quad x > 0$$

where $\varphi\downarrow$, $\varphi:(0, 1) \to \mathbb{R}^+$ and satisfies $\varphi(xy) \le C\ \varphi(x)\ \varphi(y)$. His result may then be formulated as follows:

<u>Theorem 1.10.</u> If $1 \le p < \infty$, then $\|T_\varphi f\|_{\Lambda_p(u)} \le C\|f\|_{\Lambda_p(u)}$, if and only if for each $r > 0$

$$\int_r^\infty u(x) \left(\int_0^{r/x} \varphi(t)dt\right)^p dx \le A \int_0^r u(x)dx.$$

While this result reduces to Theorem 1.9 with $\varphi \equiv 1$, it also shows with $\varphi(x) = (1-x)^{\alpha-1}$, $0 < \alpha \le 1$, a one weight characterization, of the boundedness of the Riemann–Liouville fractional integral operator on $\Lambda_p(u)$ (c.f. [20], [31] and the previous remarks).

The result of Stepanov yields a two weight characterization for a wide range of indices for the Hardy operator:

<u>Theorem 1.11.</u> ([32]). Necessary and sufficient conditions for

$$\left(\int_0^\infty (Pf)(x)^q\ u(x)dx\right)^{1/q} \le C\left(\int_0^\infty v(x)f(x)^p dx\right)^{1/p},$$

$0 < p, q < \infty$ to be satisfied for all non-negative decreasing f is that

(a) for $1 < p \le q < \infty$

$$A_0 \equiv \sup_{r>0} \left(\int_0^r u(t)dt\right)^{1/q} \left(\int_0^r v(x)dx\right)^{-1/p} < \infty$$

and

$$A_1 \equiv \sup_{r>0} \left(\int_r^\infty t^{-q} u(t)dt\right)^{1/q} \left(\int_0^r t^{p'} V(t)^{-p'} v(t)dt\right)^{1/p'} < \infty,$$

where $V(t) = \int_0^t v$, holds.

(b) For $0 < q < p < \infty$, $1/r = 1/q - 1/p$, $p > 1$,

$$B_0 \equiv \left\{\int_0^\infty \left[\left(\int_0^t u(x)dx\right)^{1/p} \left(\int_0^t v(x)dx\right)^{-1/p}\right]^r u(t)dt\right\}^{1/r} < \infty$$

and

$$B_1 \equiv \left\{\int_0^\infty \left[\left(\int_t^\infty x^{-q} u(x)dx\right)^{1/q} \left(\int_0^t x^{p'} V(x)^{-p'} v(x)dx\right)^{-1/q'}\right]^r t^{p'} V(t)^{-p'} v(t)dt\right\}^{1/r} < \infty$$

holds.

(c) For $0 < p \le q < \infty$, $0 < p < 1$, that $A_0 < \infty$ and

$$A_1^{\#} \equiv \sup_{r>0} r\left(\int_r^\infty x^{-q} u(x)dx\right)^{1/q} \left(\int_0^r v(x)dx\right)^{-1/p} < \infty .$$

It should be noted here that the index range in (c) shows that for decreasing f the mapping properties of the Hardy operator are fundamentally different than those for arbitrary f. In the index range $1 < p, q < \infty$, these results can also be obtained from Sawyer's reverse Hölder inequality ([17]):

$$\sup_{g \geq 0, g\downarrow} \frac{\int_0^\infty gf}{\left(\int_0^\infty g^p v\right)^{1/p}} \approx \left\{\int_0^\infty \left(\int_0^x f\right)^{p'} V(x)^{-p'} v(x)dx\right\}^{1/p'} + \frac{\int_0^\infty f}{\left(\int_0^\infty v\right)^{1/p}}$$

where as before $V(x) = \int_0^x v$. This results is very useful in establishing mapping properties of more general integral operators. Thus, if

$$(Tf)(x) = \int_0^\infty k(x, y)f(y)dy, \quad k(x, y) \geq 0,$$

then (with $\int_0^\infty v = \infty$) the above estimates imply that

$$\left(\int_0^\infty (Tg)(x)^q u(x)dx\right)^{1/q} \leq C\left(\int_0^\infty g(x)^p v(x)dx\right)^{1/p}$$

holds for all $g\downarrow$ and non-negative, if and only if T^*, the dual of T satisfies

$$\left[\int_0^\infty \left(\int_0^x T^*f\right)^{p'} V(x)^{-p'} v(x)dx\right]^{1/p'} \leq C\left(\int_0^\infty f(x)^{q'} u(x)^{1-q'} dx\right)^{1/q'} \qquad (1.11)$$

for all $f \geq 0$.

Now if $T = P$, the Hardy averaging operator, then

$$\int_0^x T^*f(t)dt = \int_0^x\left(\int_t^\infty \frac{f(y)}{y}\, dy\right)dt = \int_0^x\int_t^x \frac{f(y)}{y}\, dydt + \int_0^x\left(\int_x^\infty \frac{f(y)}{y}\, dy\right)dt$$

$$= \int_0^x f(y)dy + x\int_x^\infty \frac{f(y)}{y}\, dy$$

so that for $1 < p, q < \infty$, Theorem 1.11 follows from (1.11) and Theorem 1.1.

It should be noted that Sawyer's result also permits one to characterize weights for which the Hilbert transform and Riesz potential is bounded from $\Lambda_p(v)$ to $\Lambda_q(u)$, $1 \leq p, q \leq \infty$ (See e.g. [17]).

Let L denote the Laplace transform

$$(Lf)(x) = \int_0^\infty e^{-xt}\, f(t)dt \qquad x > 0,$$

then there is a simple characterization of weights for which this operator defined on decreasing functions is bounded from L^p_v to L^q_u. Since this result does not seem to follow from Sawyer's work we give the simple argument next.

Theorem 1.12. Let f be a non–negative decreasing function and $1 < p \le q < \infty$, then

$$\left(\int_0^\infty u(x)|x^{-1}(Lf)(x^{-1})|^q dx\right)^{1/q} \le C\left(\int_0^\infty v(x)f(x)^p dx\right)^{1/p} \qquad (1.12)$$

and if and only if A_0 and A_1 of Theorem 1.11 are finite.

Proof. By the second mean value theorem and the fact that f is decreasing

$$(Lf)(x) = \int_0^{1/x} e^{-xy} f(y)dy + \int_{1/x}^\infty e^{-xy} f(y)dy \le \int_0^{1/x} f(y)dy + e^{-1}f(1/x)/x$$

$$\le (1 + e^{-1}) \int_0^{1/x} f(y)dy .$$

But then by Theorem 1.11 the sufficiency part follows.

For the converse, assume first that $V(\infty) = \infty$ (recall $V(x) = \int_0^x v$) and define

$$f_r(s) = \left(\int_s^\infty x^{p'} V(x)^{-p'-1} v(x)dx\right)^{1/p} \chi_{(0,r)}(s), \quad r > 0 \text{ fixed.}$$

Then $f_r\downarrow$ and substituting into (1.12) we obtain

$$C\left(\int_0^r v(x) \left[\int_x^r y^{p'} V(y)^{-p'-1} v(y)dy\right] dx\right)^{1/p} = C\left(\left[\int_0^r y^{p'} V(y)^{-p'-1} v(y)\left(\int_0^y v(x)dx\right) dy\right)^{1/p}$$

$$= C\left(\int_0^r y^{p'} V(y)^{-p'} v(y)dy\right)^{1/p} \geq \left(\int_r^\infty u(x)x^{-q} \left|\int_0^r e^{-y/x} \left(\int_y^r t^{p'} V(t)^{-p'-1} v(t)dt\right]^{1/p} dy\right|^q dx\right)^{1/q}$$

$$\geq e^{-1} \left[\int_r^\infty u(x)x^{-q} \left|\int_0^r \left(\int_y^r t^{p'} V(t)^{-p'-1} v(t)dt\right)^{1/p} dy\right|^q dx\right]^{1/q}.$$

But the inner integral is not smaller than

$$\int_0^r y^{p'/p} \left(\int_y^r V(t)^{-p'-1} v(t)dt\right)^{1/p} dy = p \int_0^r y^{p'-1} \int_y^r \left[\int_s^r V^{-p'-1} v\, dx\right]^{-1/p'} V(s)^{-p'-1} v(s)ds dy$$

$$= p \int_0^r V(s)^{-p'-1} v(s) \left(\int_0^s y^{p'-1} dy\right) \left(\int_s^r V(\alpha)^{-p'-1} v(\alpha)d\alpha\right)^{-1/p'} ds$$

$$\geq (p-1) \int_0^r s^{p'} \left(\int_s^\infty V(\alpha)^{-p'-1} v(\alpha)d\alpha\right)^{-1/p'} V(s)^{-p'-1} v(s)ds$$

$$= (p-1) \int_0^r s^{p'} V(s)^{-p'-1} v(s) \left(\frac{V(s)^{-p'}}{p'}\right)^{-1/p'} ds = (p-1)(p')^{1/p'} \int_0^r s^{p'} V(s)^{-p'} v(s)ds.$$

Substituting, we obtain

$$C\left(\int_0^r y^{p'} V(y)^{-p'} v(y)dy\right)^{1/p} \geq e^{-1}(p-1)(p')^{1/p'} \left(\int_r^\infty u(x)x^{-q} dx\right)^{1/q} \left(\int_0^r s^{p'} v(s)\, V(s)^{-p'} ds\right)$$

from which $A_1 < \infty$ follows.

On the other hand, if $f = \chi_{(0,r)}$ is substituted into (1.12) then

$$C\left(\int_0^r v(x)dx\right)^{1/p} \geq \left(\int_0^\infty u(x)x^{-q}\left(\int_0^r e^{-y/x}\,dy\right)^q dx\right)^{1/q}$$

$$= \left(\int_0^\infty u(x)\,[1 - e^{-r/x}]^q dx\right)^{1/q} \geq \left(1 - e^{-1}\right)\left(\int_0^r u(x)dx\right)^{1/q}$$

and this implies $A_0 < \infty$, which completes the proof.

We noted before that for $p, q < 1$ and $f \geq 0$ the inequality

$$\left(\int_0^\infty f(t)^p\, v(t)dt\right)^{1/p} \leq C\left(\int_0^\infty u(x)\left(\int_0^x f(t)dt\right)^q dx\right)^{1/q} \tag{1.13}$$

is possible ([3]). However for most other values of the indices the following lemma, which I owe to Eric Sawyer, shows that (1.13) cannot hold.

Lemma 1.13. If (1.13) holds for $0 < q < \infty$, $p > 1$ and all $f \geq 0$, then $v \equiv 0$ a.e. in any interval (r, ∞) such that

$$\int_r^\infty u(t)dt < \infty.$$

Conversely, for such trivial weights, (1.13) clearly holds.

Proof. Let $F = \min(1, v^{-1/p})$ and $f_{r,s}(t) = \chi_{(r,s)}(t)F(t)$, where $0 < r < s < \infty$. Let $f = f_{r,s}$ in (1.13), then

$$\left(\int_r^s F(t)^p v(t)dt\right)^{1/p} \le C\left(\int_r^\infty u(x)dx\right)^{1/q} \left(\int_r^s F(t)dt\right) \le C(s-r)\left(\int_r^\infty u(x)dx\right)^{1/q}.$$

Therefore

$$\left[\frac{1}{s-r}\int_r^s [\min(v(t), 1)]dt\right]^{1/p} \le C(s-r)^{1/p'} \left(\int_r^\infty u(x)dx\right)^{1/q}$$

and as $s - r \to 0$ and (r, s) shrinks to a Lebesgue point x of $\min(v, 1)$ we obtain $\min(v(x), 1) \le 0$ if

$$\int_{x-\varepsilon}^\infty u(x)dx < \infty, \quad \varepsilon > 0.$$

This proves the lemma.

In light of this lemma it is perhaps surprising that for decreasing f a weighted inequality of the form (1.13) can hold.

Theorem 1.14. Let $f\downarrow$ and non-negative. If $1 \le q \le p < \infty$, then (1.13) holds if and only if for each $r > 0$

$$\left(\int_0^r v(x)dx\right)^{1/p} \le C\left(\int_0^r u(x)dx + \int_r^\infty (r/x)^q u(x)dx\right)^{1/q}.$$

Proof. For $p = q$ this result is due to Neugebauer [23] and the case $q < p$ follows from it easily.

2. <u>Weighted Fourier inequalities.</u> The first L^p–estimates involving power weights appeared in the early and mid 1930's with the work of Hardy–Littlewood–Paley–Titchmarsh and Pitt. Although many others extended and generalized these early results, it was in 1978 when Muckenhoupt ([22]) formally posed the problem of characterizing weight functions u and v for which the inequality

$$\|\hat{f}\|_{q,u} \le C\|f\|_{p,v} \tag{2.1}$$

holds for all $f \in L^1$. This problem has been studied by various authors, including Muckenhoupt, and the general result may be formulated in the following theorem ([4]):

<u>Theorem 2.1.</u> Suppose $0 < p, q < \infty$, $p \ge 1$ and u, v are non–negative even functions, such that $u\downarrow$ and $1/v\downarrow$ on $(0, \infty)$. Then

(i) for $1 \le p \le q < \infty$, (2.1) is equivalent to the condition

$$\sup_{r>0} \left(\int_0^r u(t)dt\right)^{1/q} \left(\int_0^{1/r} v(t)^{1-p'}dt\right)^{1/p'} < \infty \tag{2.2}$$

(ii) for $0 < q < p < \infty$, $p \ge 1$, the two conditions

$$\int_0^\infty \left[\left(\int_0^{1/x} u(t)dt\right)^{1/q} \left(\int_0^x v(t)^{1-p'}dt\right)^{1/q'}\right]^r v(x)^{1-p'}dx < \infty$$

and

$$\int_0^\infty \Big[\Big(\int_{1/x}^\infty t^{-q/2}\, u(t)dt\Big)^{1/q} \Big(\int_x^\infty t^{-p'/2} v(t)^{1-p'} dt\Big)^{1/q'}\Big]^r x^{-p'/2} v(x)^{1-p'} dx < \infty,$$

where $1/r = 1/q - 1/p$, are sufficient for (2.1).

Remark 2.2.

(i) The condition that u and v are even, nor the monotonicity conditions imposed on u and v are required in the sufficiency part of Theorem 2.1. One requires only that in (2.2) and in the integral conditions of (ii) that u and $1/v$ are replaced by u^* respectively $(1/v)^*$, their decreasing rearrangements.

(ii) If $0 < q \le p < 1$, weighted Fourier transform inequalities are still possible – however then the domain space of the operator must be replaced by the boundary values of functions in weighted (atomic) Hardy spaces (c.f. [11], [12]).

In the study of weighted norm inequalities for singular integral operators, the A_p–weights are the most effective weights and have been studied intensely during the last twenty years ([11], [34]). Recall that a non–negative locally integrable function w on $\mathbb{R}^n$ belongs to the A_p–weight class, $1 < p < \infty$, if there is a constant $C > 0$, such that for all n–balls $B \subset \mathbb{R}^n$ with volume $|B|$

$$\Big(\frac{1}{|B|}\int_B w(x)dx\Big)^{1/p} \Big(\frac{1}{|B|}\int_B w(x)^{1-p'} dx\Big)^{1/p'} \le C < \infty.$$

Similarly we say $w \in A_p^*$, $1 < p < \infty$, if

$$\sup_{I \subset J} \left(\frac{1}{|I|} \int_I w(x)dx \right)^{1/p} \left(\frac{1}{|I|} \int_I w(x)^{1-p'} dx \right)^{1/p'} < \infty,$$

where J denotes the collection of intervals $I = [a_1, b_1] \times \dots \times [a_n, b_n]$ in $\mathbb{R}^n$ and $|I|$ its Lebesgue measure.

It is obvious that $A_p = A_p^*$, if $n = 1$, but in general $A_p^* \subset A_p$. In fact if $w(x) = |x|^\alpha$, then $w \in A_p$, if and only if $-n < \alpha < n(p-1)$, and if

$$w(x) = \left(\sum_{i=1}^{n} |x_i| \right)^\alpha$$

then $w \in A_p^*$ implies $-1 < \alpha < p - 1$. As mentioned, these weight classes are typical Hilbert transform – singular integral criteria, so it may be somewhat surprising that they are also important in establishing weighted Fourier transforms norm inequalities.

The next part of this section describes Fourier inequalities with A_p–weights, while in the second, Fourier inequalities of functions in certain moment subspaces are discussed.

<u>2.1. Fourier transform inequalities with A_p–weights.</u> In the sequel we require the following lemma:

<u>Lemma 2.3.</u>

(a) If $w \in A_p$, $1 < p < \infty$, then

$$\int_{|x| \geq r} |x|^{-np} w(x)dx \leq C\, r^{-np} \int_{|x| \geq r} w(x)dx \tag{2.3}$$

(b) If w is radial and as radial function increasing, then $w \in A_p$ if and only if $w^{\varepsilon} \in A_{1+\varepsilon(p-1)}$, $\varepsilon > 0$.

The first part of the lemma is well known and may be found in [11] or [34]. Indeed for $n = 1$ this is essentially (1.10) of Theorem 1.9. That $w \in A_p$ implies $w^{\varepsilon} = A_{1+\varepsilon(p-1)}$ is Hölders inequality for $0 < \varepsilon < 1$. The details for $\varepsilon \geq 1$ and the converse may be found in [15].

The lemma permits us now to prove the following weighted Paley–Titchmarsh theorem:

Theorem 2.4. Suppose w is radial and as radial function increasing. Let $1 < p \leq q \leq p'$, then

$$\Big[\int_{\mathbb{R}^n} |\hat{f}(x)|^q |x|^{n(q/p'-1)} w(1/|x|)^{q/p} dx\Big]^{1/q} \leq C\Big[\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\Big]^{1/p}$$

if and only if $w \in A_p$.

For $n = 1$ the proof of this result may be found in [5] and the general case (with different proof) in [15]. Note that for $q = p'$ this is a weighted Hansdorff–Young inequality.

We sketch the sufficiency part of the proof. If $p = 2$, then by Lemma 2.3 (a) it follows that (the n–dimensional analogue of) (2.2) holds with $p = q = 2$ and $v = w$ and $u(x) = w(1/|x|)$. Therefore by Theorem 2.1

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^2 w(1/|x|)dx \leq C \int_{\mathbb{R}^n} |f(x)|^2 w(x)dx. \tag{2.4}$$

Now for any $u \geq 0$ define

$$a(x) = \int_0^{|x|} u(1/t)t^{n-1}dt \quad \text{and} \quad \mu(E) = \int_E a(x)^{-2}u(1/|x|)dx,\ E \subset \mathbb{R}^n\backslash\{0\}.$$

then

$$\mu(\{x \in \mathbb{R}^n: |a(x)\hat{f}(x)| > \lambda\}) \le \mu(\{x \in \mathbb{R}^n: |a(x)|\,\|f\|_1 > \lambda\})$$

$$= \int_{\{x: a(x) > \lambda/\|f\|_1\}} a(x)^{-2}u(1/|x|)dx = \int_\Sigma d\sigma \int_{\{t>0: a(t) > \lambda/\|f\|_1\}} t^{n-1}a(t)^{-2}u(1/t)dt$$

$$= C \int_{a^{-1}(\lambda/\|f\|_1)}^{\infty} a^{-2}(t)d\,a(t) = C\|f\|_1/\lambda,$$

where $a^{-1}(y) = \inf\{x > 0: a(x) > y\}$ so that $a(a^{-1}(h)) = y$. Now let $u = w^{p'-1}$ and replace f by f/u in this estimate, then

$$\mu(\{x \in \mathbb{R}^n: |a(x)(f/u)\hat{}(x)| > \lambda\}) \le C\|f\|_{1,1/u}/\lambda$$

which is a weighted "weak" (1,1) estimate. But Lemma 2.3(b) shows that $w \in A_p$, if and only if $u = w^{p'-1} \in A_{1+(p'-1)(p-1)} = A_2$, so that by (2.4) with f replaced by f/u

$$\int_{\mathbb{R}^n} |a(x)(f/u)\hat{}(x)|^2 d\mu(x) \le C\|f\|^2_{2,1/u}.$$

This inequality together with the "weak" type (1,1) above yields (via the Marcinkiewicz interpolation theorem)

$$\int_{\mathbb{R}^n} |a(x)(f/u)\hat{}(x)|^p d\mu(x) \le C\|f\|^p_{p,1/u} \qquad 1 < p < 2$$

and with f/u replaced by f

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^p \, a(x)^{p-2} u(1/|x|)dx \le C \int_{\mathbb{R}^n} |f(x)|^p \, u(x)^{p-1} dx . \qquad (2.5)$$

But since Lemma 2.3(a) implies that

$$a(x) = \int_{1/|x|}^{\infty} t^{-n-1} u(t)dt \quad = C \int_{|y|>1/|x|} |y|^{-2n} u(y)dy \le C|x|^{2n} \int_{|y|\le 1/|x|} u(y)dy$$

$$\le C|x|^n u(1/|x|),$$

substitution into the left side of (2.5) (note $p - 2 < 0$) yields

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^p |x|^{n(p-2)} u(1/|x|)^{p-1} dx \le C \int_{\mathbb{R}^n} |f(x)|^p u(x)^{p-1} dx.$$

But $u = w^{p'-1}$ so that Theorem 2.4 holds with $q = p$. The general case follows similarly.

The monotonicity conditions imposed on the weights both in Theorem 2.1 and Theorem 2.4 are highly undesirable. In [15] the condition of radial monotonicity was replaced by insisting on appropriate monotonicity condition in each variable separately. In this way one obtains the following weighted Hausdorff–Young inequality:

$$\left(\int_{\mathbb{R}^n} |\hat{f}(x)|^{p'} u(1/x)dx\right)^{1/p'} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p u(x)^{p-1} dx\right)^{1/p}, \quad 1 < p \le 2,$$

holds if and only if $u \in A_2^*$.

A detailed study of weighted Fourier estimates with weights of this type was made by Bloom, Jurkat and Sampson [7]. One of their results is the following:

Theorem 2.5. Suppose $1 < p \leq q < \infty$ and w and $1/v$ are even and decreasing in $(0, \infty)$ in each of their variables separately. If

$$u(x) = |x_1 x_2 \dots x_n|^{q-2} w(1/x)$$

is increasing in each x_i, then

$$\left(\int\limits_{\mathbb{R}^n} |\hat{f}(x)|^q \, w(x)dx\right)^{1/q} \leq C\left(\int\limits_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p}$$

if and only if

$$\left(\int\limits_{\mathbb{R}^n} |(P_n f)(x)|^q u(x)dx\right)^{1/q} \leq C\left(\int\limits_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p}$$

where

$$(P_n f)(x) = \frac{1}{|x_1 x_2 \dots x_n|} \int\limits_{|t_1| \leq |x_1|} \dots \int\limits_{|t_1| \leq |x_n|} f(t)dt$$

is the n–dimensional averaging operator.

We point out here, however, that weight characterizations for P_n: $L^p_v \to L^q_u$ are only known for $n = 1$ and $n = 2$. The problem for $n > 2$ is still open.

Although it is possible to weaken the monotonicity conditions on weights in these last theorems somewhat ([4]) its total deletion – and hence a complete characterization of weights – have not been proved. There are also several abstractions of Theorem 2.1 in the case $1 < p \leq q < \infty$ to locally compact connected Abelian groups and certain locally compact totally disconnected groups. However, either the weight conditions given are only sufficient, or additional restrictions on the weights – corresponding to monotonicity – must be obtained to prove characterizations. (c.f. [17], [24], [25]).

If instead of imposing monotonicity conditions on the weight functions one restricts the functions f, then much more may be said. In fact in that case we have the following characterization:

Theorem 2.6. Let f be a non-negative even function in L^p_v, $v \geq 0$, $1 \leq p < \infty$, such that f is decreasing in $(0, \infty)$ and $f(\infty) = 0$. Then

$$\int_{-\infty}^{\infty} |\hat{f}(1/x)/x|^p v(x)dx \leq C \int_{-\infty}^{\infty} |f(x)|^p v(x)dx,$$

if and only if (2.3) holds with $n = 1$ and $w = v$.

It is easy to see that $v(x) = |x|^{-\varepsilon}$, $0 < \varepsilon < 1$ satisfies (2.3) with $n = p = 1$ and $w = v$.

2.2. Fourier transform inequalities of functions with vanishing moments. The last theorem suggests that if restrictions are imposed on the functions f rather than the weights, then one might expect also that a larger class of weights can be generated for which weighted Fourier norm inequalities hold. For example, if $\hat{f}(0) = 0$, then

$$|\hat{f}(x)| = \left|\int_{-\infty}^{\infty} (e^{-ixt} - 1)f(t)dt\right| = \left|\int_{|xt|\le 1} + \int_{|xt|>1}\right|$$

$$\le 2\int_{|t|\le 1/|x|} |xt||f(t)|dt + 2\int_{|t|>1/|x|} |f(t)|dt = 2\left\{\int_{|y|\ge|x|} |y^{-3}f(1/y)|dy + \int_{|y|\le|x|} |y^{-2}f(1/y)|dy\right\}.$$

But these integrals are essentially the Hardy operator and its dual of $|y^{-3}f(1/y)|$ and $|y^{-2}f(1/y)|$ respectively. Therefore, applying the weighted form of the Hardy operator and its dual one is led to the following result of Sadosky and Wheeden [26]:

<u>Theorem 2.7.</u> If $1 < p < \infty$ and $w \in A_p$, then

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^p w(1/x) \frac{dx}{|x|^2} \le C \int_{-\infty}^{\infty} |f(x)|^p |x|^p w(x)dx \tag{2.6}$$

for all f satisfying $\hat{f}(0) = 0$.

Compare this with Theorem 2.4 taking $q = p$ and $n = 1$.

If $w(x) = 1/|x|$, then $w \notin A_p$, $p > 1$ and one might ask if (2.6) with this weight and $\hat{f}(0) = 0$ is satisfied? The answer is no, and in fact, no norm inequality of the form

$$\left(\int_{-\infty}^{\infty} |\hat{f}(x)|^q u(x)dx\right)^{1/q} \le C \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{p-1}dx\right)^{1/p},$$

$1 < p < \infty$, $0 < q \le \infty$, $u \not\equiv 0$, can hold for f satisfying $\hat{f}(0) = 0$ ([26]).

The idea to consider Schwartz functions f satisfying

$$\int_{-\infty}^{\infty} f(x)x^j dx = 0, \quad j = 0,1,2,\dots ;$$

and whose Fourier transforms have compact support away from the origin has been exploited in [6] and [26] to prove Fourier norm inequalities for substantially larger weight classes. These subspaces are known to be dense in weighted L^p spaces ([6], [21]) and the Fourier transform has a natural extension in these spaces. It is also clear that corresponding results carry over to mixed indices.

To apply Fourier inequalities to obtain uncertainty relations of the Heisenberg–Weyl type, or Fourier restriction theorems it is desirable to replace the weights in the range space by general measures. This in turn requires corresponding measure estimates for the Hardy operator and its dual. Such results – indeed characterizations – were proved by Sinnamon [29], and the corresponding measure weighted Fourier inequalities on moment subspaces were studied in [6]. We state here only the one dimensional case:

Theorem 2.8. Suppose

(i) $v \in L^1_{loc}(\mathbb{R})$, $v > 0$ a.e. and μ a positive measure.

(ii) $1 < p \le q < \infty$, $v^{1-p'} \in L^1_{loc}(\mathbb{R}\backslash[-y,y])$ for each $y > 0$.

(iii) $$B_1 \equiv \sup_{y>0} \left\{\int_{|t|<y} |t|^q d\mu(t)\right\}^{1/q} \left\{\int_{|t|<1/y} |t|^{p'} v(t)^{1-p'} dt\right\}^{1/p'}$$

and

$$B_2 \equiv \sup_{y>0} \left\{\int_{|t|>y} d\mu(t)\right\}^{1/q} \left\{\int_{|t|>1/y} v(t)^{1-p'} dt\right\}^{1/p'}$$

are both finite.

(iv) $f \in L^p_v(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\hat{f}(0) = 0$ with supp f compact.

Then there is a constant $C > 0$ (depending on B_1 and B_2) such that

$$\|\hat{f}\|_{q,\mu} \leq C\|f\|_{p,v}.$$

For example if $v(t) = |t|^{1+\varepsilon}$, $0 < \varepsilon < 2$, $\mu = \Sigma' |n|^{-1-\epsilon} \delta_n$, where δ is the Dirac distribution and $p = q = 2$, then all conditions of this theorem are satisfied and one obtains

$$\Sigma' \frac{|\hat{f}(n)|}{|n|^{1+\varepsilon}} \leq 2^5(\pi B_1 + B_2)^2 \int_{\mathbb{R}} |f(t)|^2 |t|^{1+\varepsilon} dt.$$

(For the details see [6, Ex. 2.5(c)].)

Some progress has been made to characterize the Fourier transform in weighted Lorentz spaces although a complete solution seems not to be available. We conclude this section with a characterization of a special Fourier transform in a very special setting due to Braverman.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X(w)$ a random variable. In this context, the Calderon $L^{p,q}$ spaces are then defined by $X \in L^{p,q}$, if and only if

$$\|X\|^*_{p,q} = \left(\int_0^\infty (P[|X| > x])^{q/p} \, dx^q\right)^{1/q} < \infty,$$

where $0 < p, q < \infty$. Let $F(y) = P[X > y]$ and the characteristic function (Fourier transform) be defined by

$$\hat{F}(y) = \int_{-\infty}^{\infty} e^{ity} d\, F(y).$$

The result alluded to is then the following:

Theorem 2.9. If $1 < p < 2$, $1 \le q < \infty$, then

$$\left\{ \int_0^{\infty} \left[1 - \mathrm{Re}\left(t \int_0^{1/t} \hat{F}(y) dy \right) \right]^{q/p} dt^q \right\}^{1/q} \approx \|X\|^*_{p,q} .$$

3. Some applications. In 1974, Zygmund ([35]) considered the two dimensional Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}^2} e^{-i\, x \cdot y} f(y) dy \qquad x \in \mathbb{R}^2$$

and estimated the restriction of the Fourier transform on circles $|x| = \rho$, in terms of the L^p-norm of f. His result states that if $f \in L^p(\mathbb{R}^2)$, $1 \le p < 4/3$, then $\hat{f}(x)$ exists a.e. on $|x| = \rho$ and

$$\left(\int_{|x|=\rho} |\hat{f}(x)|^q ds \right)^{1/q} \le A_p \rho^{1/p'} \|f\|_p \qquad \textbf{(3.1)}$$

for $q = p'/3$. Moreover, the result is sharp in the sense that if $p = 4/3$ then inequality (3.1) fails.

At almost the same time Sjölin [30] provided an extension theorem – that is, a dual Fourier estimate of measures carried by smooth curves in $\mathbb{R}^2$, which, when restricted to curves of constant curvature (circles) implies Zygmund's results. Since then, a vast literature on the subject evolved with many significant applications.

The proof of the result of Zygmund (and also Sjölin's) utilizes among other things, a duality argument, the Hausdorff–Young inequality and an L^p–estimate of the Riemann–Liouville fractional integral operator. But all these components permit generalizations and weighted extensions. It is not surprising therefore, that these restriction–extension theorems have weighted extensions in $\mathbb{R}^2$. These generalizations, proved in [9] have the form

$$\|\hat{f}\,|K|^{\delta}\|_{L^r(\gamma,ds)} \le A\|f\|_{L^p_v(\mathbb{R}^2)} \tag{3.2}$$

where γ is a smooth plane curve in $\mathbb{R}^2$ of curvature K and arclength measure ds.

A very specific case of the generalizations of Zygmund's estimate (3.1) is given here ([9, Corollary 2, with $\lambda = 0$]).

<u>Proposition 3.1.</u> If $4/3 < q < \infty$, $1 < p < 6q/(3q+2)$ and

$$\max[0,\, 2(1/p' - 1/(3q))] \le \alpha < 2/p',$$

then for $1 \le r \le q$,

$$\left(\int_{|x|=\rho} |\hat{f}(x)|^r ds\right)^{1/r} \le C\rho^{1/r+\alpha-2/p'}\|f\|_{p,\alpha}\,.$$

In particular, if $q = p'/3$, then $1 < p < 4/3$ and then taking $\alpha = 0$ in Proposition 3.1, we obtain Zygmund's result.

A special case of the generalization of Sjölin's theorem has the same form. In fact under the hypotheses Proposition 3.1 one obtains (3.2) with $v(x) = |x|^{\alpha}$, $\delta > 1/(3q)$ and $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $t \in [ab]$. Moreover (3.2) holds also if $\delta = 1/(3q)$, provided $K(t) \geq 0$. (c.f [9, Cor. 1]).

It should be noted that these results are strictly two dimensional since the weights in the appropriate Fourier inequalities which lead to these results are especially adapted to the geometry of the curve. Higher dimensional weighted spherical restriction theorems can be obtained directly from the n–dimensional measure weighted Fourier inequalities. That is the n–dimensional version of Theorem 2.8.

Theorem 3.2. ([6, Theorem 5.3]) Let $v \in L^1_{loc}(\mathbb{R}^n)$, $v > 0$, a.e., radial and $1 < p \leq q < \infty$. Assume that $v^{1-p'} \in L^1_{loc}(\mathbb{R}^n \backslash B(0,y))$ for each $y > 0$ and

$$C(p,q,n,\rho) \equiv A(n,p,q)\rho^{\frac{n-1}{q}} \left\{\rho\left(\int_0^{1/(\rho\pi)} r^{n-1+p'} v(r)^{1-p'} dr\right)^{1/p'} + \left(\int_{1/(\rho\pi)}^{\infty} r^{n-1} v(r)^{1-p'} dr\right)^{1/p'}\right\}, \ \rho > 0$$

Then for all $f \in M_0(n) \cap L^p_v(\mathbb{R}^n)$

$$\left(\int_{\Sigma_{n-1}(\rho)} |\hat{f}(x)|^q d\sigma_{n-1}\right)^{1/q} \leq C(p,q,n,\rho)\|f\|_{p,v}.$$

We conclude with an example of the uncertainty inequality. Given $(x_0, y_0) \in \mathbb{R}^2$, then the classical uncertainty inequality states that for all $f \in \mathcal{S}(\mathbb{R})$

$$\int_{\mathbb{R}} |f(x)|^2 dx \leq 4\pi\left(\int_{\mathbb{R}} |(x - x_0)f(x)|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} |(y - y_0)\hat{f}(y)|^2 dy\right)^{1/2}$$

with equality if $f(x) = c\, e^{-s(x-x_0)^2} e^{2\pi i x y_0}$, $s > 0$, c a complex constant.

There are numerous variants and generalizations of this inequality with applications in science and engineering. It is not difficult to see that the change $f(x) \to f(x + x_0)e^{-2\pi i x y_0}$ shows that one may take without loss of generality $x_0 = y_0 = 0$ in this inequality.

Recall from Section 1, if

$$(Qf)(x) = \int_1^\infty f(xt)\frac{dt}{t} \quad x \in \mathbb{R}^n$$

and $(Q^*f)(x) = -x\cdot\nabla f(x)$, $x \in \mathbb{R}^n$, then $QQ^*f(x) = f(x)$. Instead of applying Theorem 1.2(i) with ρ replaced by p' and $q = p'$, $v(x) = u(x) = |x|^{-p'}$ we prove the result directly in the following lemma:

<u>Lemma 3.3.</u> If $f \in C_0^1(\mathbb{R}^n)$, $1 < p' < n$, then

$$\left(\int_{\mathbb{R}^n} \left|\frac{f(x)}{x}\right|^{p'} dx\right)^{1/p'} \le \frac{p'}{n-p'} \left(\int_{\mathbb{R}^n} \left|\frac{x\cdot\nabla f}{x}\right|^{p'} dx\right)^{1/p'}$$

<u>Proof.</u>

$$\left[\int_{\mathbb{R}^n} \left|\frac{f(x)}{x}\right|^{p'} dx\right]^{1/p'} = \left[\int_{\mathbb{R}^n} \left|\frac{QQ^*f(x)}{x}\right|^{p'} dx\right]^{1/p'} = \left[\int_{\mathbb{R}^n} |x|^{-p'} \left|\int_1^\infty (Q^*f)(xs)\frac{ds}{s}\right|^{p'} dx\right]^{1/p'}$$

$$= \left\{\int_\Sigma\int_0^\infty t^{n-p'-1}\left|\int_1^\infty (Q^*f)(t\sigma s)\frac{ds}{s}\right|^{p'} dt d\sigma\right\}^{1/p'} = \left\{\int_\Sigma\int_0^\infty t^{n-p'-1}\left|\int_t^\infty (Q^*f)(y\sigma)\frac{dy}{y}\right|^{p'} dt d\sigma\right\}^{1/p'}$$

$$\le \frac{p'}{n-p'}\left\{\int_\Sigma\int_0^\infty |Q^*(t\sigma)|^{p'} t^{n-p'-1} dt d\sigma\right\}^{1/p'} = \frac{p'}{n-p'}\left(\int_{\mathbb{R}^n}\left|\frac{x\cdot\vec\nabla f(x)}{x}\right|^{p'} dx\right)^{1/p'}$$

where the last inequality is the classical conjugate Hardy inequality.

<u>Theorem 3.4.</u> If $f \in C_0^1(\mathbb{R}^n)$, $1 < p \le 2$ and $n > p'$, then

$$\int_{\mathbb{R}^n} |f(x)|^2 dx \le C\left(\int_{\mathbb{R}^n} |xf(x)|^p dx\right)^{1/p} \left(\int_{\mathbb{R}^n} |y\hat{f}(y)|^p dy\right)^{1/p},$$

where

<u>Proof.</u> Hölder's inequality and Lemma 3.3 yield

$$\int_{\mathbb{R}} |f(x)|^2 dx \le \left(\int_{\mathbb{R}^n} |xf(x)|^p dx\right)^{1/p} \left(\int_{\mathbb{R}^n} |\tfrac{f(x)}{x}|^{p'} dx\right)^{1/p'}$$

$$\le \frac{p'}{n-p'}\left(\int_{\mathbb{R}^n} |xf(x)|^p dx\right)^{1/p} \left(\int_{\mathbb{R}^n} |\tfrac{x\cdot\vec{\nabla}f(x)}{x}|^{p'} dx\right)^{1/p'}$$

By Minkowski's inequality, and writing $\frac{\partial f}{\partial x_i} = \check{h}$, $i = 1,2,\ldots,n$; it follows that the right integral product is

$$\left(\int_{\mathbb{R}^n} \left|\sum_{i=1}^n \frac{x_i}{|x|}\frac{\partial f}{\partial x_i}(x)\right|^{p'} dx\right)^{1/p'} \le \sum_{i=1}^n \left(\int_{\mathbb{R}^n} \left|\frac{x_i}{|x|}\check{h}(x)\right|^{p'} dx\right)^{1/p'}$$

$$= \sum_{i=1}^n \left(\int_{\mathbb{R}^n} |(R_i h)\check{}(x)|^{p'} dx\right)^{1/p'} \le \sum_{i=1}^n \left(\int_{\mathbb{R}^n} |(R_i h)(x)|^p dx\right)^{1/p}$$

$$\le C\sum_{i=1}^n \left(\int_{\mathbb{R}^n} |h(x)|^p dx\right)^{1/p} \le n^{1/p'} C\left[\sum_{i=1}^n \int_{\mathbb{R}^n} |h(x)|^p dx\right]^{1/p}$$

$$= Cn^{1/p'}\Big[\int_{\mathbb{R}^n} \sum_{i=1}^{n} |\Big(\frac{\partial f}{\partial x_i}\Big)^{\wedge}(x)|^p dx\Big]^{1/p} = Cn^{1/p'}\Big[\int_{\mathbb{R}^n} \Sigma|y_i \hat{f}(y)|^p dy\Big]^{1/p}$$

$$\leq cn^{1/2}\Big[\int_{\mathbb{R}^n} |\hat{f}(y)y|^p dy\Big]^{1/p},$$

where we applied the Hausdorff–Young inequality, the L^p–boundedness of the Riesz transform R and Hölder's inequality twice. Substituting we get the result.

It is clear that on using Theorem 1.2(i) and the well known weighted L^p–boundedness of the Riesz transform ([11][34]) a corresponding weighted inequality can be obtained . This unweighted form is however curious since the classical $n = 1$ – case does not follow from it.

For additional recent application of weighted Fourier inequalities we refer to [4] [15] were generalizations of the Paley–Wiener theorems were given and Laplace representations in weighted Bergman spaces were established.

References

[1] K.F. Andersen and H.P. Heinig, Weighted norm inequalities for certain integral operators, SIAM J. Math. Anal. 14, (1983), 834–844.

[2] M.A. Ariño and B. Muckenhoupt, Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for non–increasing functions, (Preprint).

[3] P.R. Beesack and H.P. Heinig, Hardy's inequalities with indices less than 1; Proc. Amer. Math. Soc. 83(3)(1981) 532–536.

[4] J.J. Benedetto, H.P. Heinig and R. Johnson, Weighted Hardy spaces and the Laplace transform II, Math. Nachr. 132, (1987), 29–55.

[5] ————; Fourier inequalities with A_p–weights, Proc. Conf. Oberwolfach 1986, General Inequalities 5, Int. Ser. Num. Math. 80 Birkhäuser, Basel (1987), 217–232.

[6] J.J. Benedetto and H.P. Heinig, Fourier transform inequalities with measure weights, Advances in Math. (to appear).

[7] S. Bloom, W.B. Jurkat and G. Sampson, Two weighted (L^p, L^q) estimates for the Fourier transform, (Preprint).

[8] M. Sh. Braverman, On a class of operators, (Preprint).

[9] C. Carton–Lebrun and H.P. Heinig, Weighted extensions and restriction theorems for the Fourier transform, (Preprint).

[10] J.A. Cochran and C.S. Lee, Inequalities related to Hardy's and Heinig's, Math. Proc., Cambr. Phil. Soc. 96 (1984), 1–7.

[11] G. Garcia–Cuerva and J.L. Rubio de Francia, Weighted norm inequalities and related topics, North Holland, Amsterdam 1985.

[12] H.P. Heinig, Fourier operators on weighted Hardy spaces, Math. Proc. Cambr. Phi. Soc. 101 (1987) 103–121.

[13] ———; Some extensions of Hardy's inequality, SIAM J. Math. Anal. 6 (1975) 698–713.

[14] ———; Weighted norm inequalities for certain integral operators II, Proc. A.M.S. 95, (1985), 387–395.

[15] H.P. Heinig and G.J.Sinnamon, Fourier inequalities and integral representations of functions in weighted Bergman spaces over tube domains, Indiana Univ. Math. J. 38 (3), (1989), 603–628.

[16] E. Hernandez, Factorization and extrapolation of pairs of weights, (Preprint).

[17] J.A. Hogan, Weighted norm inequalities for the Fourier transform on connected locally compact groups, Pac. J. Math. 131 (2), (1988), 277–290.

[18] A. Kufner and B. Opic, Hardy's inequalities, generalizations and applications, Longman–Pittman 1990 (to appear).

[19] E.R. Love,Some inequalities for geometric means; ISNM 80, General Inequalities 5; W. Walter ed., Birkhäuser 1987, 83–93

[20] F.J. Martin–Reyes and E.Sawyer, Weighted Inequalities for Riemann–Liouville fractional integrals of order one and greater (to appear).

[21] B. Muckenhoupt, R. Wheeden and W.S. Young, L^2–multipliers with power weights, Advances in Math. 49 (1983), 170–216.

[22] B. Muckenhoupt, Weighted norm inequalities for classical operators, Proc. Symp, Pure Math. 35, AMS Providence, R.I. (1979), 69–83.

[23] C.J. Neugebauer, Weighted norm inequalities for averaging operator of monotone functions (Preprint).

[24] C.W. Onneweer, Weak L^p–spaces and weighted norm inequalities for the Fourier transform on locally compact Vilenkin groups, Proc. Anal. Conf. Singapore (1986), 191–201, North Holland Math. Studies 150, Amsterdam, N.Y. 1988.

[25] T.S. Quek, Weighted norm inequalities for the Fourier transform on certain totally disconnected groups, Proc. A.M.S. 101 (1), (1987), 113–121.

[26] C. Sadosky and R.L. Wheeden, Some weighted norm inequalities for the Fourier transform of functions with vanishing moments, Trans. Amer. Math. Soc. 300, (1987) (2), 521–533.

[27] E. Sawyer, Boundedness of classical operators are classical Lorentz spaces, (Preprint).

[28] G. Sinnamon, A weighted gradient inequality, Proc. Roy. Soc. Edinburgh, 111 A (1989), 329–335.

[29] ———; Operators on Lebesgue spaces with general measures, Ph.D. thesis, McMaster Univ. 1987.

[30] P. Sjölin, Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in $\mathbb{R}^2$. Studia Math. 51, (1974), 169–182.

[31] V.D. Stepanov, Two weighted estimates for the Riemann–Liouville integrals, Česk. Akad. Věd; Mat. Ustav. Preprint 339 (1988) 1–28.

[32] ———; The two weighted Hardy's inequality for non–increasing functions, (Preprint).

[33] C.R. Stromberg, Introduction to classical real analysis, Wadsworth, Belmont, Calif. 1981.

[34] A. Torchinsky, Real variable methods in harmonic analysis; Acad. Press, N.Y. 1986.

[35] A. Zygmund, On Fourier coefficients and transforms of functions of two variables, Studia Math. 50 (1974) 189–210.

Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, Canada L8S 4K1

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