TRANSMISSION OF CONVERGENCE

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Abstract. If $E(f) = \{x : \limsup f \star \mu_j(x) > \liminf f \star \mu_j(x)\}$, we examine the type of convergence of g_k to f so that $|E(g_k)| \leq M$, $k = 1, 2, \ldots$, implies $|E(f)| \leq M$.

1. Introduction

Let $\{\mu_j\}_{j \geq 1}$ be positive Borel measures on $\mathbb{R}^n$ with $\operatorname{supp} \mu_j \subset K$, K compact, and normalized so that $\mu_j(\mathbb{R}^n) = 1$, $j = 1, 2, \ldots$. For $f : \mathbb{R}^n \to [0, \infty]$ — throughout all functions will be *non-negative* — let

$$E(f) = \{x : \limsup f \star \mu_j(x) > \liminf f \star \mu_j(x)\},$$

the exceptional set for convergence of $\{f \star \mu_j(x)\}$, where

$$f \star \mu_j(x) = \int_{\mathbb{R}^n} f(x + y)\, d\mu_j(y).$$

The problem we wish to study in this note is to estimate $|E(f)|$ with $\{|E(g_k)|\}$ for appropriate approximations of $\{g_k\}$ to f, i.e., when are the convergence properties of $\{g_k \star \mu_j\}_{j \geq 1}$ transmitted to $\{f \star \mu_j\}_{j \geq 1}$ as $k \to \infty$? If we can control the maximal operator

$$Mf(x) = \sup_j f \star \mu_j(x)$$

then it is well known that $g_k \to f$ in L^p is enough. In fact:

1991 *Mathematics Subject Classification.* 42B25.

Key words and phrases. measure; convergence; maximal operator; minimal operator.

Assume that

$$Mf(x) < \infty \text{ on a set of positive measure for every } f \in L^p,$$
$$g_k \to f_0 \text{ in } L^p.$$

Then $|E(f_0)| \leq \liminf |E(g_k)|$.

To see this, first observe that by Proposition 1 in [6, p. 441],

$$|\{x : Mf(x) > y\}| \leq \frac{A}{y^p} \|f\|_p^p,$$

that is, Mf is of weak type (p,p). Write $E(f_0) = \bigcup E_i$, where

$$E_i = \{x : \limsup f_0 \star \mu_j(x) - \liminf f_0 \star \mu_j(x) > 1/i\}.$$

For g_k fixed, after adding and subtracting $\limsup g_k \star \mu_j(x) - \liminf g_k \star \mu_j(x)$, we get

$$\begin{aligned} E_i &\subset \{x : 2 \limsup [f_0 \star \mu_j(x) - g_k \star \mu_j(x)] > 1/(2i)\} \cup E(g_k) \\ &\subset \{x : M(|f_0 - g_k|)(x) > 1/(4i)\} \cup E(g_k) \end{aligned}$$

and thus

$$|E_i| \leq A(4i)^p \|f_0 - g_k\|_p^p + |E(g_k)|.$$

Thus $|E_i| \leq \liminf |E(g_k)|$, and hence $|E(f_0)| \leq \liminf |E(g_k)|$. $\square$

R e m a r k. To obtain the last displayed inequality one only needs that

$$|\{x : M(|f_0 - g_k|)(x) > y\}| \leq \frac{c}{y^p} \|f_0 - g_k\|_p^p \tag{1}$$

with c independent of k and $y > 0$. We shall use this remark later.

The hypothesis on the maximal operator Mf is not satisfied in many interesting situations. For example, if $d\mu_j = \dfrac{\chi_{R_j}}{|R_j|} dx$, where the R_j's are oriented rectangles containing the origin and $|R_j| \to 0$, then Mf is not of weak type $(1,1)$; or, if $d\mu_j = \dfrac{\chi_{R_j}}{|R_j|} dx$, where the R_j's are arbitrary rectangles containing the origin with $|R_j| \to 0$, then Mf is not of weak type (p,p) for any p, $1 \leq p < \infty$. Other examples where the maximal operator cannot be controlled for a given p are given by measures μ_j singular with respect to Lebesgue measure, e.g., $Mf(x) = \sup_{t>0} f \star d\sigma_t(x)$, maximal averages over surfaces. For further details we refer the reader to [6, Ch. 11]. It is precisely the cases where Mf is too large which interest us and which we wish to examine. To this end we need an A_s^*-condition and the minimal operator.

We write for $0 < s < \infty$ and $\phi : \mathbb{R}^n \to [0, \infty]$,

$$A_s^*(\phi) = \sup_{j,x} \phi \star \mu_j(x) \cdot \left(\frac{1}{\phi^s} \star \mu_j(x) \right)^{1/s} .$$

We observe that in the special case where $d\mu = \dfrac{\chi_Q}{|Q|}\, dx$, Q an arbitrary cube with $0 \in Q$, if $A_s^*(\phi) < \infty$, then ϕ is in the Muckenhoupt A_p-weight class, $p = 1 + 1/s$ (see [4], [5]).

The *minimal operator of order* s is defined by

$$m_s f(x) = \inf_j \big(f^s \star \mu_j(x)\big)^{1/s}.$$

The behavior of m_s is much better than that of M. We shall show that under the sole assumption (4) of Theorem 1 below, $|\{x : m_s f(x) < 1/y\}| \le (c_q/y^q)\|1/f\|_q^q$ for any q, $0 < q < \infty$ (see Section 2), and under the stronger assumption (5) of Theorem 2 below, m even satisfies a distributional inequality $|\{x : mf(x) < 1/y\}| \le c_1|\{x : f(x) < c_2/y\}|$ (see Section 3). Moreover, if M is of weak type (p_0, p_0) for some p_0, $1 \le p_0 < \infty$, then $\|1/m_s f\|_q \le c\|1/f\|_q$ for any q, $0 < q < \infty$ (see Section 6).

Hölder's inequality shows that $m_{s'} f \le m_{s''} f$ if $s' \le s''$, and we write $m_\infty f = \lim\limits_{s\to\infty} m_s f$.

One of our main results is:

Theorem 1. *Assume that* $0 < p, r, s < \infty$. *If*

$$\text{either} \quad \frac{1}{g_k} \to \frac{1}{f_0} \ \text{in } L^p \quad \text{or} \quad g_k \to f_0 \ \text{in } L^p, \tag{2}$$

$$A_s^*(|g_k - f_0|) \le c < \infty, \quad k = 1, 2, \ldots, \tag{3}$$

$$m_\infty f(x) > 0 \ \text{on a set of positive measure for every } f,\ \frac{1}{f} \in L^r(\mathbb{R}^n), \tag{4}$$

then $|E(f_0)| \le \liminf |E(g_k)|$.

R e m a r k. In the special cases where $d\mu_j = \dfrac{\chi_{E_j}}{|E_j|}\, dx$, the differentiation of the integral case, or $d\mu_j = \phi_{\varepsilon_j} dx$, the approximate identity case, this type of problem was already examined in [2], [3] with a more restrictive hypothesis.

In Section 5 we shall examine a version of Theorem 1 where the L^p-convergence in (2) is relaxed and (4) is strengthened. In particular, let $\{\nu_j\}$ be another sequence of Borel measures on $\mathbb{R}^n$ with $\nu_j(\mathbb{R}^n) = 1$ and $\operatorname{supp} \nu_j \subset K$, $j = 1, 2, \dots$. Now let

$$A_1^*(\phi) = \sup_{j,x} \phi \star \mu_j(x) \cdot \frac{1}{\phi} \star \nu_j(x),$$
$$mf(x) \equiv m_\nu f(x) = \inf_j f \star \nu_j(x).$$

Finally, let $L_0 = \big\{f : |\{x : f(x) < 1\}| < \infty\big\}$. Note that, if $1/f \in L^r$, then $f \in L_0$.

Theorem 2. *If*

$$g_k \to f_0 \text{ a.e. as } k \to \infty,$$
$$A_1^*(|g_k - f_0|) \le c < \infty, \quad k = 1, 2, \dots,$$
$$mf(x) > 0 \text{ on a set of positive measure for every } f \in L_0, \tag{5}$$

then $|E(f_0)| \le \liminf |E(g_k)|$.

The following is an example illustrating the type of convergence in Theorem 2. Let $\alpha_k \searrow 0$ with $\alpha_k/\alpha_{n+1} \le c < \infty$. If $\{g_k\}$ satisfies $\alpha_{n+1} \le |g_k(x) - f_0(x)| \le \alpha_k$ for each x, then $A_1^*(|g_k - f_0|) \le c$.

The proofs of Theorems 1, 2 will be given in Sections 4, 5. In Sections 2 and 3 we examine weak-type and distributional inequalities for the minimal operator which we need for the proof of Theorems 1 and 2. Section 6 contains some remarks and variants of these Theorems.

2. Weak-type inequalities

This section is devoted to showing that the condition (4) of Theorem 1 implies a weak-type inequality for $m_s f$.

Definition. We say that m_s is of weak type (r, r) on E (with constant A) if for every f with $\operatorname{supp} 1/f \subset E$,

$$|\{x : m_s f(x) < 1/y\}| \le \frac{A}{y^r} \Big\| \frac{1}{f} \Big\|_r^r$$

where A is independent of $y > 0$ and f.

We let $Q = [0, 1)^n$ and we let Q^* be a cube containing $Q + K$, where K is the common support of $\{\mu_j\}$.

Lemma 3. *Let $j \in \mathbb{Z}^n$ and let $Q_j^* = Q^* + j$. If m_s is of weak type (r, r) on Q_j^* with constant A, then m_s is of weak type (r, r) on any other Q_i^* with the same constant.*

Proof. This follows from translation invariance. □

Lemma 4. *If m_s is of weak type (r, r) on Q^*, then m_s is of weak type (r, r) on $\mathbb{R}^n$.*

Proof. Let $1/f \in L^r(\mathbb{R}^n)$, and let $Q_j = Q + j$, $Q_j^* = Q^* + j$, $j \in \mathbb{Z}^n$. If $f_j = f/\chi_{Q_j^*}$, then from Lemma 3,

$$|\{x : m_s f_j(x) < 1/y\}| \le \frac{A}{y^r} \Big\| \frac{1}{f_j} \Big\|_r^r.$$

Note that $\sum \chi_{Q_j^*} \le N < \infty$.

If $x_0 \in \mathbb{R}^n$, then x_0 is in a unique Q_j and thus $m_s f(x_0) = m_s f_j(x_0)$. Hence

$$\{x : m_s f(x) < 1/y\} \subset \bigcup_j \{x : m_s f_j(x) < 1/y\}$$

from which

$$|\{x : m_s f(x) < 1/y\}| \le \frac{A}{y^r} \sum_j \Big\| \frac{1}{f_j} \Big\|_r^r.$$

Since

$$\frac{1}{N} \sum_j \frac{1}{f_j(x)^r} = \frac{1}{N} \sum_j \frac{\chi_{Q_j^*}(x)}{f(x)^r} \le \frac{1}{f(x)^r},$$

we obtain

$$|\{x : m_s f(x) < 1/y\}| \le \frac{NA}{y^r} \Big\| \frac{1}{f} \Big\|_r^r$$

and the proof is complete. □

Lemma 5. *Assume that m_s is not of weak type (r, r) on $\mathbb{R}^n$. Then there exists $F : \mathbb{R}^n \to [0, \infty]$ such that $m_s F(x) = 0$ for a.e. x, and $1/F \in L^r(\mathbb{R}^n)$.*

Proof. From Lemma 4 we know that m_s is not of weak type (r, r) on Q^*. Hence, for every $k \in \mathbb{N}$ there is $y_k > 0$ and g_k such that $1/g_k \in L^r(\mathbb{R}^n)$, $\operatorname{supp} 1/g_k \subset Q^*$ and

$$|\{x : m_s g_k(x) < 1/y_k\}| > \frac{2^k}{y_k^r} \Big\| \frac{1}{g_k} \Big\|_r^r.$$

If B^* is a cube containing $Q^* - K$, then

$$|\{x : m_s g_k(x) < 1/y_k\}| = |\{x \in B^* : m_s g_k(x) < 1/y_k\}|,$$

since $m_s g_k(x) = \infty$, $x \notin B^*$.

Let $g'_k = y_k g_k/k$. Then

$$|\{x \in B^* : m_s g'_k(x) < 1/k\}| \geq \frac{2^k}{k^r} \Big\| \frac{1}{g'_k} \Big\|_r^r.$$

Hence $|B^*|/\|1/g'_k\|_r^r \to \infty$ and so $\|1/g'_k\|_r^r \to 0$. By passing to a subsequence, we may assume that $\sum \|1/g'_k\|_r^r < \infty$. We can now find a sequence $\{f_k\}$, $f_k = g'_{j_k}$ with possible repetitions, and $r_k \to 0$ such that, if $E_k = \{x \in B^* : m_s f_k(x) < r_k\}$, then $\sum |E_k| = \infty$ and $\sum \|1/f_k\|_r^r < \infty$.

By the Lemma in [6, p. 442], there is $\{x_k\} \subset \mathbb{R}^n$ such that, if $F_k = E_k + x_k$, then

$$\limsup F_k = \bigcap_{k\geq 1} \bigcup_{j \geq k} F_j = \mathbb{R}^n$$

except for a set of measure zero. Now we let $\tilde{f}_k(x) = f_k(x - x_k)$ and

$$F(x) = \inf_k \tilde{f}_k(x).$$

Then $m_s F(x) \leq \inf_k m_s \tilde{f}_k(x)$, and so $m_s F(x) \leq r_k$, $x \in F_k$. Therefore, $m_s F(x) = 0$ for a.e. x. Since

$$\frac{1}{F(x)^r} = \sup_k \frac{1}{\tilde{f}_k(x)^r} \leq \sum_k \frac{1}{\tilde{f}_k(x)^r},$$

we see that $1/F \in L^r(\mathbb{R})$. □

Remark. It may be of interest to have an example where m_s is not of weak type (r, r). Let $D = \{x_j\}_{j \geq 1}$ be a countable dense subset of $B = \{x : |x| \leq 1\}$, and let $\mu_j = \delta(x_j)$. If $f \in C(\mathbb{R}^n)$ with $f(0) = 0$ and $1/f \in L^r(\mathbb{R}^n)$, then for $x \in B$, $m_s f(x) = 0$. Consequently, m_s is not of weak type (r, r) on $\mathbb{R}^n$.

Lemma 6. *Assume that $0 < q, r < \infty$ and that m_s is of weak type (r, r) on $\mathbb{R}^n$. Then m_s is of weak type (q, q) on $\mathbb{R}^n$.*

Proof. By Lemma 5 it suffices to show that $m_s f(x) > 0$ on a set of positive measure for every f, $1/f \in L^q(\mathbb{R}^n)$. If $q < r$, then $1/f^{q/r} \in L^r(\mathbb{R}^n)$ and by Hölder's inequality

$$(f^{qs/r} \star \mu_j(x))^{1/s} \leq (f^s \star \mu_j(x))^{q/(rs)}.$$

Assume now that $q > r$. If Q and Q^* are as above, then for $x \in Q$, $m_s f(x) = m_s(f/\chi_{Q^*})(x)$ and $\chi_{Q^*}/f \in L^r(\mathbb{R}^n)$. □

Lemma 7. *If $0 < r, s, t < \infty$ and m_t is of weak type (r, r) on $\mathbb{R}^n$, then m_s is of weak type (r, r) on $\mathbb{R}^n$.*

Proof.

$$|\{x : m_s f(x) < 1/y\}| = |\{x : [m_t(f^{s/t})(x)]^{t/s} < 1/y\}| \le \frac{A}{y^{sr/t}} \Big\| \frac{1}{f} \Big\|_{sr/t}^{sr/t}.$$

Lemma 6 completes the proof. □

We are now ready to prove our main weak-type inequality result.

Theorem 8. *Assume that $0 < q, r, s < \infty$ and $m_\infty f(x) > 0$ on a set of positive measure for every f, $1/f \in L^r(\mathbb{R}^n)$. Then m_s is of weak type (q, q) on $\mathbb{R}^n$.*

Proof. If we deny the conclusion, then by Lemma 6, m_s is not of weak type (r, r) on $\mathbb{R}^n$. Hence, by Lemma 7, m_j is not of weak type (r, r) on $\mathbb{R}^n$ for every $j \in \mathbb{N}$. By Lemma 5, we have for each $j \in \mathbb{N}$ a function $F_j : \mathbb{R}^n \to [0, \infty]$ such that $m_j F_j(x) = 0$ for a.e. x and $1/F_j \in L^r(\mathbb{R}^n)$. We now choose $0 < \alpha_j < \infty$ such that $\sum \alpha_j \|1/F_j\|_r^r < \infty$.

Let $F = \inf_j F_j/\alpha_j^{1/r}$. Then for every j and for a.e. x,

$$m_j F(x) \le \alpha_j^{-1/r} m_j F_j(x) = 0.$$

Hence $m_\infty F(x) = 0$ for a.e. x. Since

$$\frac{1}{F^r} = \sup_j \frac{\alpha_j}{F_j^r} \le \sum_j \frac{\alpha_j}{F_j^r},$$

we see that $1/F \in L^r(\mathbb{R}^n)$. This contradicts our hypothesis. □

Remarks. (i) The proofs of Lemmas 4 and 5 proceed along the lines of the proof of Proposition 1 in [6, p. 441] for the maximal operator.

(ii) We do not know whether the hypothesis of Theorem 8 implies the strong-type inequality

$$\int_{\mathbb{R}^n} \frac{dx}{m_s f^q} \le c_q \int_{\mathbb{R}^n} \frac{dx}{f^q}.$$

In Section 6 we shall present a condition which will give us this strong-type inequality. We shall also make a comment in Section 6 concerning the weak-type (q, q) constant of m_s.

(iii) For the example in the remark after Lemma 5 where m_s was not of weak type (r, r), the above theorem gives a function F such that $m_\infty F(x) = 0$ for a.e. x and $1/F \in L^r(\mathbb{R}^n)$.

3. Distributional inequalities

This section is similar to the previous one and deals with a distributional inequality for $mf(x) \equiv m_\nu f(x) = \inf_j f \star \nu_j(x)$, where $\{\nu_j\}$ is a sequence of Borel measures on $\mathbb{R}^n$ with $\nu_j(\mathbb{R}^n) = 1$ and $\operatorname{supp} \nu_j \subset K$, $j = 1, 2, \ldots$, where K is a compact subset of $\mathbb{R}^n$.

Definition. We say that m satisfies a distributional inequality on E with constants c_1, c_2 if, and only if,

$$|\{x : mf(x) < 1/y\}| \leq c_1 |\{x : f(x) < c_2/y\}|$$

for every $f : \mathbb{R}^n \to [0, \infty]$ with $\operatorname{supp} 1/f \subset E$.

We use the same notation as in Section 2 for Q, Q^*, Q_j and Q_j^*. From translation invariance, if m satisfies a distributional inequality on Q_j^* with constants c_1, c_2, then the same is true on any other Q_i^*.

Lemma 9. *If m satisfies a distributional inequality on Q^*, then m satisfies a distributional inequality on $\mathbb{R}^n$.*

P r o o f. Let $f_j = f/\chi_{Q_j^*}$. By the above observation, there are constants c_1, c_2 such that

$$|\{x : mf_j(x) < 1/y\}| \leq c_1 |\{x : f_j(x) < c_2/y\}|,$$

where c_1, c_2 are independent of f and j. Note that $\sum \chi_{Q_j^*} \leq N < \infty$. As in Lemma 4,

$$|\{x : mf(x) < 1/y\}| \leq c_1 \sum_j |\{x : f_j(x) < c_2/y\}|.$$

Since $E_j = \{x : f_j(x) < c_2/y\} = \{x : f(x) < c_2/y\} \cap Q_j^*$, we see that $\sum \chi_{E_j} \leq \chi_{\{f < c_2/y\}}(x) \cdot N$ and thus $\sum |E_j| \leq N |\{x : f(x) < c_2/y\}|$. □

For the next result we recall that L_0 is the class of $f : \mathbb{R}^n \to [0, \infty]$ such that $|\{x : f(x) < 1\}| < \infty$.

Theorem 10. *Assume that $mf(x) > 0$ on a set of positive measure for every $f \in L_0$. Then m satisfies a distributional inequality on $\mathbb{R}^n$.*

P r o o f. If we deny the conclusion, then by Lemma 9, m does not satisfy a distributional inequality on Q^*. Hence for every $k \in \mathbb{N}$, we have $y_k > 0$ and a function g_k with $\operatorname{supp} 1/g_k \in Q^*$ such that

$$L_k \equiv |\{x : mg_k(x) < 1/y_k\}| \geq 2^k |\{x : g_k(x) < c_k/y_k\}|$$

for some $c_k \to \infty$. Let $g'_k = (y_k g_k)/c_k$. Then

$$L_k = |\{x \in B^* : mg'_k(x) < 1/c_k\}| \geq 2^k |\{x : g'_k(x) < 1\}|,$$

where, as in Lemma 5, B^* is a cube containing $Q^* - K$. From this we get that $|\{x : g'_k(x) < 1\}| \to 0$, and thus we may assume that $\sum |\{x : g'_k(x) < 1\}| < \infty$. Consequently, there exist $r_k \to 0$ and $f_k = g'_j$ with possible repetitions such that, if $E_k = \{x \in B^* : mf_k(x) < r_k\}$, then $\sum |E_k| = \infty$ and $\sum |\{x : f_k(x) < 1\}| < \infty$. As in Lemma 5, we have $\{x_k\} \subset \mathbb{R}^n$ such that, if $F_k = E_k + x_k$, then

$$\limsup F_k = \mathbb{R}^n$$

except for a set of measure zero. We now set $\tilde{f}_k = f_k(x - x_k)$ and $F(x) = \inf_k \tilde{f}_k(x)$. Since $mF(x) \leq \inf_k m\tilde{f}_k(x)$ and $m\tilde{f}_k(x) \leq r_k$, $x \in F_k$, we see that $mF(x) = 0$ for a.e. x. Also note that $F \in L_0$ since

$$|\{x : F(x) < 1\}| \leq \sum_j |\{x : \tilde{f}_k(x) < 1\}| < \infty.$$

This contradicts our hypothesis, and the proof is complete. □

R e m a r k. It may be of interest to give an example of m satisfying the hypothesis of Theorem 10. Let $mf(x) = \inf_{x \in Q} \frac{1}{|Q|} \int_Q f$, where Q is a cube. Let $f \in L_0$ and let $E = \{x : f(x) \geq 1\}$. Then $|E| = \infty$. We claim that $mf(x) > 0$ at every point of density of E. If x_0 is such a point of E, and $x_0 \in Q$, then $|E \cap Q|/|Q| \to 1$ as $|Q| \to \infty$ or 0. Hence, $\inf_j f \star \chi_{Q_j}/|Q_j|$, $Q_j \subset K$, satisfies a distributional inequality. In Section 6 we remove the restriction $Q_j \subset K$ and we give a double weight generalization of this distributional inequality in $\mathbb{R}$.

4. Proof of Theorem 1

We shall first assume that $1/g_k \to 1/f_0$ in L^p and prove that

$$|E(f_0)| \leq \liminf |E(g_k)|.$$

We first observe that if $x \in E(f_0)$, then $\limsup f_0 \star \mu_j(x) > \liminf f_0 \star \mu_j(x)$ and hence $\liminf f_0 \star \mu_j(x) < \infty$. Moreover, by Theorem 8 with $s = 1$ and $q = p$, $\liminf f_0 \star \mu_j(x) > 0$ for a.e. x. Thus

$$E(f_0) = \Big\{x : \limsup \frac{1}{f_0 \star \mu_j(x)} > \liminf \frac{1}{f_0 \star \mu_j(x)}\Big\}.$$

We write $E(f_0) = \bigcup E_i$, where

$$E_i = \left\{ x : \limsup \frac{1}{f_0 \star \mu_j(x)} - \liminf \frac{1}{f_0 \star \mu_j(x)} > \frac{1}{i} \right\}.$$

We now fix g_k and observe that

$$\begin{aligned} E_i \subset \Big\{ x : \limsup &\Big(\frac{1}{f_0 \star \mu_j(x)} - \frac{1}{g_k \star \mu_j(x)} \Big) \\ &- \liminf \Big(\frac{1}{f_0 \star \mu_j(x)} - \frac{1}{g_k \star \mu_j(x)} \Big) \\ &+ \limsup \frac{1}{g_k \star \mu_j(x)} - \liminf \frac{1}{g_k \star \mu_j(x)} > \frac{1}{i} \Big\} \\ \subset \Big\{ x : 2 \limsup &\Big| \frac{1}{f_0 \star \mu_j(x)} - \frac{1}{g_k \star \mu_j(x)} \Big| > \frac{1}{2i} \Big\} \cup E(g_k). \end{aligned}$$

Let

$$A_j(x) = \left[\left(\frac{f_0 \cdot g_k}{|f_0 - g_k|} \right)^{\alpha} \star \mu_j(x) \right]^{1/\alpha}, \quad \alpha = \frac{s}{2s+1}.$$

By Hölder's inequality with exponents $p_1 = p_2 = (2s+1)/s$, $p_3 = 2s+1$, we get

$$A_j(x) \leq f_0 \star \mu_j(x) \cdot g_k \star \mu_j(x) \cdot \left(\frac{1}{|f_0 - g_k|^s} \star \mu_j(x) \right)^{1/s}$$

and hence using (3)

$$|f_0 \star \mu_j(x) - g_k \star \mu_j(x)| A_j(x) \leq c f_0 \star \mu_j(x) \cdot g_k \star \mu_j(x).$$

Consequently, if $0 < f_0 \star \mu_j(x) \cdot g_k \star \mu_j(x) < \infty$, then

$$\left| \frac{1}{f_0 \star \mu_j(x)} - \frac{1}{g_k \star \mu_j(x)} \right| \leq c A_j(x)^{-1} \leq c \left[m_\alpha \left(\frac{f_0 g_k}{|f_0 - g_k|} \right)(x) \right]^{-1}. \tag{6}$$

If both $f_0 \star \mu_j(x)$ and $g_k \star \mu_j(x)$ are infinite, then (6) is obvious, and, if, say, $g_k \star \mu_j(x) = \infty$ and $f_0 \star \mu_j(x) < \infty$, then $g_k \star \mu_j(x) = |f_0 - g_k| \star \mu_j(x)$ and thus

$$\frac{A_j(x)}{f_0 \star \mu_j(x)} \leq c$$

and (6) follows. Hence

$$
\begin{aligned}
|E_i| &\le \left|\left\{x : m_\alpha\left(\frac{f_0 g_k}{|f_0-g_k|}\right)(x) < 4ci\right\}\right| + |E(g_k)| \\
&\le C i^p \left\|\frac{|f_0-g_k|}{f_0 g_k}\right\|_p^p + |E(g_k)|.
\end{aligned}
$$

The second inequality follows from Theorem 8 with m_α in place of m_s and $q = p$. This gives us $|E_i| \le \liminf |E(g_k)|$ and thus $|E(f_0)| \le \liminf |E(g_k)|$.

We shall now prove that $|E(f_0)| \le \liminf |E(g_k)|$ assuming that $g_k \to f_0$ in L^p. This is where we use (1) in the remark in the introduction. Since by condition (3),

$$
|g_k - f_0| \star \mu_j(x) \cdot \left(\frac{1}{|g_k-f_0|^s} \star \mu_j(x)\right)^{1/s} \le c < \infty
$$

we have

$$
M(|f_0-g_k|)(x) \le c\left[m_s\left(\frac{1}{|f_0-g_k|}\right)(x)\right]^{-1},
$$

and hence by Theorem 8 with $q = p$,

$$
\begin{aligned}
|\{x : M(|f_0-g_k|)(x) > y\}| &\le |\{x : m_s(1/(|f_0-g_k|))(x) < c/y\}| \\
&\le \frac{A}{y^p}\|f_0-g_k\|_p^p.
\end{aligned}
$$

Since the constant A does not depend on k or $y > 0$, the inequality (1) completes the proof. □

Remark. The hypothesis (4) of Theorem 1 requires that $m_\infty f(x) > 0$ on a set of positive measure for every f, $1/f \in L^p$. In the special case

$$
m_s f(x) = \inf_{x\in Q}\left(\frac{1}{|Q|}\int_Q f^s\right)^{1/s},
$$

where Q is a cube in $\mathbb{R}^n$, $m_\infty f(x)$ can be readily estimated.

If $1/f \in L^p(\mathbb{R}^n)$ for some p, $0 < p < \infty$, then $m_\infty f(x) \ge f(x)$ for a.e. x. If in addition $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $m_\infty f(x) = f(x)$ for a.e. x.

Proof. Let x_0 be a point of approximate continuity of f and $f(x_0) > 0$. For $\lambda < f(x_0)$, the set $E_\lambda = \{x : f(x) > \lambda\}$ has x_0 as a point of density.

Since $1/f \in L^p$, there is $N > 0$ such that

$$|Q| \geq N \quad \text{implies} \quad \left(\frac{1}{|Q|}\int_Q \frac{1}{f^p}\right)^{1/p} < \frac{1}{f(x_0)}.$$

Since $1 = \frac{1}{|Q|}\int_Q f^\alpha f^{-\alpha}$, for any $s > 0$, by Hölder's inequality with $\alpha = sp/(s+p)$ and exponents $r = (s+p)/p$, $r' = (s+p)/s'$, we have

$$\frac{1}{\left(\frac{1}{|Q|}\int_Q f^s\right)^{1/s}} \leq \left(\frac{1}{|Q|}\int_Q \frac{1}{f^p}\right)^{1/p}.$$

Thus for $x_0 \in Q$ and $|Q| \geq N$

$$f(x_0) < \left(\frac{1}{|Q|}\int_Q f^s\right)^{1/s}.$$

Consider now those Q with $x_0 \in Q$ and $|Q| < N$. If

$$c = \inf_{\substack{x_0 \in Q \\ |Q| < N}} \frac{|Q \cap E_\lambda|}{|Q|},$$

then $c > 0$. For such Q's we have

$$\left(\frac{1}{|Q|}\int_Q f^s\right)^{1/s} \geq \left(\frac{1}{|Q|}\int_{Q\cap E_\lambda} \lambda^s\right)^{1/s} \geq \lambda c^{1/s}.$$

Consequently, $m_s f(x_0) \geq \lambda c^{1/s}$ and hence $m_\infty f(x_0) \geq \lambda$.

If f is also locally integrable, then $m_s f(x) \leq f(x)$ for a.e. x and thus $m_\infty f(x) \leq f(x)$ for a.e. x. □

5. Proof of Theorem 2

By Theorem 10, we have the distributional inequality

$$|\{x : mf(x) < 1/y\}| \leq c_1 |\{x : f(x) < c_2/y\}|.$$

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, and $\Phi(t) = \int_0^t \phi$, then

$$\int_{\mathbb{R}^n} \Phi(1/mf(x))\, dx \leq c_1 \int_{\mathbb{R}^n} \Phi(c_2/f(x))\, dx.$$

To see this, multiply the distributional inequality by $\phi(y)$ and integrate in y from 0 to ∞. Write $|\{x : mf(x) < 1/y\}| = \int \chi_E(x,y)\,dx$, where $E = \{(x,y) : mf(x) < 1/y\}$. Interchange the order of integration to obtain the left-hand side of the integral inequality. Below we apply the integral inequality with $\phi(\tau) = (1+\tau^2)^{-1}$, $\Phi(t) = \tan^{-1} t$.

For N a positive integer, let $B_N = \{x : |x| \le N\}$ and let $B_{N'}$ be a ball containing $B_N + K$. Then $E_N(f_0) \equiv E(f_0) \cap B_N \subset E(f_0^{N'})$, where $f_0^{N'} = f_0\chi_{B_{N'}}$. The A_1^*-condition implies that

$$\sup_j \sup_{x\in B_N} \big|g_k^{N'} - f_0^{N'}\big| \star \mu_j(x) \cdot \left(\frac{1}{|g_k^{N'} - f_0^{N'}|} \star \nu_j(x)\right) \le c < \infty.$$

Thus for $x \in B_N$,

$$M\big(|g_k^{N'} - f_0^{N'}|\big)(x) \le \frac{c}{m\big(1/|g_k^{N'} - f_0^{N'}|\big)(x)}.$$

If $E_{iN} = E_i \cap B_N$, then as before

$$|E_{iN}| \le \left|\left\{x : \frac{1}{4ci} < \frac{1}{m\big(1/|g_k^{N'} - f_0^{N'}|\big)(x)}\right\}\right| + |E(g_k)|.$$

Thus

$$\begin{aligned}|E_{iN}| &\le \left|\left\{x : \Phi\Big(\frac{1}{m\big(1/|g_k^{N'} - f_0^{N'}|\big)(x)}\Big) > \Phi\Big(\frac{1}{4ci}\Big)\right\}\right| + |E(g_k)|\\ &\le \frac{c_1}{\Phi(1/(4ci))}\int_{\mathbb{R}^n} \Phi\big(c_2\big|g_k^{N'} - f_0^{N'}\big|(x)\big)\,dx + |E(g_k)|.\end{aligned}$$

The integrand goes to zero as $k \to \infty$ for a.e. x and is bounded by $\chi_{B_{N'}} \cdot \pi/2$. The Lebesgue Dominated Convergence Theorem shows that

$$|E_{iN}| \le \liminf |E(g_k)|.$$

To complete the proof, let $N \to \infty$ and then $i \to \infty$. □

As an illustration, let $d\nu_j = \dfrac{\chi_{Q_j}}{|Q_j|}\,dx$, where $Q_j \subset K$, $j = 1, 2, \ldots$, and let $\{\mu_j\}_{j\ge1}$ be a sequence of Borel measures with $\mu_j(\mathbb{R}^n) = 1$ and $\operatorname{supp}\mu_j \subset K$ for every j.

Corollary. *If $g_k \to f_0$ a.e. and $A_1^*(|g_k - f_0|) \le c < \infty$ for each k, then $|E(f_0)| \le \liminf |E(g_k)|$.*

P r o o f. By the remark after Theorem 10, $mf(x) = \inf_j f\star\nu_j(x)$ is positive on a set of positive measure for every $f \in L_0$. □

6. Concluding remarks

In this final section we shall make some comments about the results in the previous sections and point out some generalizations.

1. We included the case $1/g_k \to 1/f_0$ in Theorem 1 because in the differentiation of the integral case ($d\mu_j = \dfrac{\chi_{E_j}}{|E_j|}\,dx$, $E_j \to 0$) $E(f) = E(F)$, $F(x) = f(x) + \mathrm{e}^{|x|}$ and $1/F \in L^p(\mathbb{R}^n)$.

2. It may be of interest to compare mf with $M(1/f)$. Since

$$1 = \left(f^{1/2} \cdot f^{-1/2} \star \mu_j(x)\right)^2 \le f \star \mu_j(x) \cdot \frac{1}{f} \star \mu_j(x),$$

we get

$$\frac{1}{mf(x)} \le M(1/f)(x).$$

If $M(1/f)(x) < \infty$ on a set of positive measure (the hypothesis of Proposition 1 in [6, p. 441]), then $mf(x) > 0$ on this set. The converse is not true. An example is the strong differentiation of the integral of $f \in L^1(\mathbb{R}^n)$, $n > 1$. The stronger finiteness assumption on the maximal function allows us to replace the weak-type inequality of Theorem 8 by a strong-type inequality.

Assume that $0 < s < \infty$ *and* $1 \le p_0 < \infty$. *If* $Mf(x) < \infty$ *on a set of positive measure for every* $f \in L^{p_0}(\mathbb{R}^n)$, *then*

$$\int_{\mathbb{R}^n} \frac{dx}{(m_s f)^q} \le c \int_{\mathbb{R}^n} \frac{dx}{f^q}, \quad 0 < q < \infty,$$

with the constant c *independent of* f *and* q.

P r o o f. Since, by Proposition 1 in [6, p. 441],

$$|\{x : Mf(x) > y\}| \le \frac{c}{y^{p_0}} \|f\|_{p_0}^{p_0}$$

and since $\|Mf\|_\infty \le \|f\|_\infty$, we can apply the Marcinkiewicz Interpolation Theorem and get $\|Mf\|_p^p \le c_p \|f\|_p^p$ for $p_0 < p < \infty$. Fix $p_0 < p_1 < \infty$. If $0 < \sigma < 1$, then

$$1 = \left(f^\sigma \cdot f^{-\sigma} \star \mu_j(x)\right)^{1/\sigma} \le f \star \mu_j(x) \cdot \left(\frac{1}{f^r} \star \mu_j(x)\right)^{1/r},$$

where $r = \sigma/(1-\sigma)$ or $\sigma = r/(1+r)$. Hence

$$\frac{1}{mf(x)} \le M(1/f^r)(x)^{1/r} \quad \text{or} \quad \frac{1}{m_s f(x)} \le M(1/f^\varrho)(x)^{1/\varrho},\ \varrho = rs.$$

Now let $0 < q < \infty$ and let $\varrho = q/p_1$. Then

$$\int_{\mathbb{R}^n} \frac{dx}{(m_s f)^q} \le \int_{\mathbb{R}^n} M(1/f^\varrho)^{q/\varrho} dx \le c_{p_1} \int_{\mathbb{R}^n} \frac{dx}{f^q}.$$

□

3. If we strengthen the A_s^*-condition, we can drop condition (4) of Theorem 1. More generally, let $\{\mu_{jx}\}_{j\ge1}$ be positive Borel measures, $x \in \mathbb{R}^n$. As before we let

$$E(f) = \Big\{x : \limsup \int_{\mathbb{R}^n} f\, d\mu_{jx} - \liminf \int_{\mathbb{R}^n} f\, d\mu_{jx} > 0\Big\}.$$

Let $B_{jx} = \{t : |t-x| \le 1/j\}$ and let $d\mu_{jx}^* = d\mu_{jx} + |B_{jx}|^{-1}\chi_{B_{jx}} dy$. Note that, if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $|E(f)| = |E^*(f)|$, where $E^*(f)$ is defined in the same way as $E(f)$ with μ_{jx} replaced by μ_{jx}^*. Finally, let $0 < s < \infty$ and

$$A_s'(\phi) = \sup_{j,x} \int_{\mathbb{R}^n} \phi\, d\mu_{jx}^* \cdot \int_{\mathbb{R}^n} \Big(\frac{1}{\phi^s}\, d\mu_{jx}^*\Big)^{1/s}.$$

Theorem 11. *Let $0 < p, s < \infty$ and let $f_0, g_k \in L^1_{\text{loc}}(\mathbb{R}^n)$, $k = 1, 2, \ldots$. If*

$$\text{either} \quad \frac{1}{g_k} \to \frac{1}{f_0} \text{ in } L^p \quad \text{or} \quad g_k \to f_0 \text{ in } L^p,$$

$$A_s'(|f_0 - g_k|) \le c < \infty,$$

then $|E(f_0)| \le \liminf |E(g_k)|$.

Proof. If $m_s^* f(x) = \inf_j \big(\int f^s\, d\mu_{jx}^*\big)^{1/s}$ and $m_s f(x) = \inf_j \big(\frac{1}{|B_{jx}|}\int_{B_{jx}} f^s\big)^{1/s}$, then $m_s^* f(x) \ge m_s f(x)$ and hence for every q, $0 < q < \infty$, using the Remark 2 above, we obtain

$$|\{x : m_s^* f(x) < 1/y\}| \le |\{x : mf_s(x) < 1/y\}| \le \frac{A}{y^q}\Big\|\frac{1}{f}\Big\|_q^q.$$

The rest of the proof is the same as that of Theorem 1. □

4. We examine now the problem of two sequences of measures $\{\mu_j\}_{j\geq 1}$ and $\{\nu_j\}_{j\geq 1}$ where $\nu_j(\mathbb{R}^n) = 1$ and $\operatorname{supp}\nu_j \subset K$, K compact, $j = 1, 2, \ldots$. We do *not* assume that the measures μ_j satisfy these two conditions. This is different from the hypothesis of Theorem 2. We define

$$A_1^*(\phi) = \sup_{j,x} \phi \star \mu_j(x) \cdot \frac{1}{\phi} \star \nu_j(x),$$
$$mf(x) = \inf_j f \star \nu_j(x).$$

The following theorem is the $s = 1$ version of Theorem 1. We could have considered $s > 1$, but for the application we have in mind $s = 1$ is sufficient.

Theorem 12. *Let $0 \leq p, r < \infty$ and assume that*

$$g_k \to f_0 \text{ in } L^p,$$
$$A_1^*(|g_k - f_0|) \leq c < \infty, \tag{7}$$
$$mf(x) > 0 \text{ on a set of positive measure for every } f,\ \frac{1}{f} \in L^r(\mathbb{R}^n). \tag{8}$$

Then $|E(f_0)| \leq \liminf |E(g_k)|$.

Proof. As before, let $Mf(x) = \sup f \star \mu_j(x)$. If

$$E_i = \{x : \limsup f_0 \star \mu_j(x) - \liminf f_0 \star \mu_j(x) > 1/i\},$$

then

$$E_i \subset \{x : M(|g_k - f_0|)(x) > 1/(4i)\} \cup E(g_k).$$

The hypothesis (7) implies

$$M(|g_k - f_0|)(x) \leq \frac{c}{m(1/|g_k - f_0|)(x)}$$

and thus

$$E_i \subset \{x : m(1/|g_k - f_0|)(x) < 4ic\} \cup E(g_k).$$

Finally, using the results from Section 2 with μ_j replaced be ν_j,

$$|E_i| \leq c(4i)^p \|g_k - f_0\|_p^p + |E(g_k)|.$$

Now let $k \to \infty$ and then $i \to \infty$. □

For example, if in Theorem 12 the measures $\{\nu_j\}$ are $d\nu_j = \dfrac{\chi_{B_j}}{|B_j|}\,dx$, where B_j are balls with center 0 and radius $r_j \to 0$, then condition (8) can be omitted. As an application, we consider the differentiability of the integral with respect to $\{E_j\}$. Here, $\{E_j\}$ is a sequence of sets with $E_j \subset \{x : |x| \le \varepsilon\}$, $j \ge j_\varepsilon$ and $|E_j| > 0$. Even if $E_j = B(x_j, r_j) = \{x : |x - x_j| \le r_j\}$ it may happen that the maximal operator $Mf(x) = \sup_j f \star \mu_j(x)$ is not of weak type (p,p) for any p, $1 \le p < \infty$ (see [2], [6]), where $d\mu_j = \dfrac{\chi_{E_j}}{|E_j|}\,dx$.

5. This remark concerns the size of the weak-type (q,q) constant of m_s in Theorem 8.

If

$$|\{x : m_s f(x) < 1/y\}| \le \frac{A_{p_0}}{y^{p_0}} \left\| \frac{1}{f} \right\|_{p_0}^{p_0},$$

then for $0 < p \le p_0$,

$$|\{x : m_s f(x) < 1/y\}| \le \frac{A_{p_0}}{y^{p}} \left\| \frac{1}{f} \right\|_{p}^{p}.$$

P r o o f. Let $\sigma > sp_0$. Then $m_\sigma f(x) \ge m_{sp_0} f(x)$ and hence

$$\begin{aligned} |\{x : m_\sigma f(x) < 1/y\}| &\le |\{x : m_{sp_0} f(x) < 1/y\}| \\ &= |\{x : m_s(f^{p_0})(x) < 1/y^{p_0}\}| \\ &\le \frac{A_{p_0}}{y^{p_0^2}} \left\| \frac{1}{f^{p_0}} \right\|_{p_0}^{p_0}. \end{aligned}$$

Hence, since $m_\sigma f = m_s(f^{\sigma/s})^{s/\sigma}$,

$$|\{x : m_s(f^{\sigma/s})(x) < 1/y^{\sigma/s}\}| \le \frac{A_{p_0}}{y^{p_0^2}} \left\| \frac{1}{f^{p_0}} \right\|_{p_0}^{p_0}.$$

Let $\phi = f^{\sigma/s}$ and $t = y^{\sigma/s}$. Then, if we choose $\sigma > sp_0$ so that $p = p_0^2 s/\sigma$, we get

$$|\{x : m_s \phi(x) < 1/t\}| \le \frac{A_{p_0}}{t^{p}} \left\| \frac{1}{\phi} \right\|_{p}^{p}.$$

$\square$

As an application, let for $0 < p < \infty$

$$d_p(g,f) = \begin{cases} \|g - f\|_p^p, & 0 < p < 1 \\ \|g - f\|_p, & p \ge 1 \end{cases}$$

be the standard metric on $L^p(\mathbb{R}^n)$. We have the following variant of Theorem 1.

Assume that $0 < r < \infty$ *and* $0 < p_k < \infty$, $k = 1, 2, \dots$. *If*

$$d_{p_k}(g_k, f_0) \to 0 \quad as \quad k \to \infty,$$
$$A_1^*(|g_k - f_0|) \le c < \infty, \ k = 1, 2, \dots,$$
$$|\{x : mf(x) > 0\}| > 0 \ for\ every\ f,\ \frac{1}{f} \in L^r,$$

then $|E(f_0)| \le \liminf |E(g_k)|$.

Proof. Let $B_N = \{x : |x| \le N\}$ and let $N' > N$ so that $B_{N'} \supset B_N + K$. We claim

$$\Big|\Big\{x \in B_N : m\Big(\frac{1}{|g_k - f_0|}\Big)(x) < 4ci\Big\}\Big| \le A_1\, 4ci\, d_{p_k}(g_k, f_0)\tau_k,$$

where A_1 is the constant above for $p = 1$ and $\tau_k = 1$ if $0 < p_k < 1$ and equals $|B_{N'}|^{1/p_k'}$, if $1 \le p_k < \infty$. We only need to consider the case $1 \le p_k < \infty$. If $g_k^{N'} = g_k \chi_{B_{N'}}$, $f_0^{N'} = f\chi_{B_{N'}}$, then

$$\begin{aligned}\Big|\Big\{x \in B_N &: m\Big(\frac{1}{|g_k - f_0|}\Big)(x) < 4ci\Big\}\Big| \\ &= \Big|\Big\{x \in B_N : m\Big(\frac{1}{|g_k^{N'} - f_0^{N'}|}\Big)(x) < 4ci\Big\}\Big| \\ &\le A_1\, 4ci\, \big\|g_k^{N'} - f_0^{N'}\big\|_1 \\ &\le A_1\, 4ci\, d_{p_k}(g_k, f_0)|B_{N'}|^{1/p_k'}.\end{aligned}$$

Hence

$$|E_{iN}| \equiv |E_i \cap B_N| \le |E(g_k)| + A_1\, 4ci\, d_{p_k}(g_k, f_0)\tau_k.$$

Finally, let k, N and i go to ∞ in this order. □

6. This remark deals with the limiting case ($s \to 0$) of Theorem 1. Let

$$\lim_{s\to 0} \left(f^s \star \mu_j(x)\right)^{1/s} = L_j(f, x), \quad m_0 f(x) = \lim_{s\to 0} m_s f(x).$$

Since $\mu_j(\mathbb{R}^n) = 1$ it is known that $L_j(f, x_0) = \exp[(\log f) \star \mu_j(x_0)]$, if $f^{s_0} \star \mu_j(x_0) < \infty$ for some s_0, $0 < s_0 < \infty$. Further, if this finiteness restriction is valid for every j, then $m_0 f(x_0) = \inf_j \exp\{(\log f) \star \mu_j(x_0)\}$. To see this, let $m_* f(x_0)$ be the right-hand side and note that by Jensen's inequality

$$\exp[(\log f) \star \mu_j(x_0)] \le \left(f^s \star \mu_j(x_0)\right)^{1/s}$$

from which $m_* f(x_0) \le m_0 f(x_0)$. For the reverse inequality, let $\lambda < m_0 f(x_0)$. Then for $j \in \mathbb{N}$ and $0 < s \le s_0$ we have $\lambda < (f^s \star \mu_j(x_0))^{1/s}$. Hence $\lambda \le \exp[(\log f) \star \mu_j(x_0)]$ and $\lambda \le m_* f(x_0)$.

We denote by

$$A_0^*(\phi) = \sup_{j,x} \phi \star \mu_j(x) \cdot L_j(1/\phi, x)$$

Note that in the special case, where $d\mu = \frac{\chi_Q}{|Q|}\,dx$, A_0^* corresponds to the Muckenhoupt weight class A_∞.

Since $m_0(1/\phi)(x) \le \inf_j L_j(1/\phi, x)$, we have, if $A_0^*(\phi) = c < \infty$, that

$$M\phi(x) \le \frac{c}{m_0(1/\phi)(x)}.$$

Theorem 13. *Let $0 < p, r < \infty$. If*

$$\text{either} \quad \frac{1}{g_k} \to \frac{1}{f_0} \text{ in } L^p \quad \text{or} \quad g_k \to f_0 \text{ in } L^p,$$
$$A_0^*(|g_k - f_0|) \le c < \infty,$$
$$m_0 f(x) > 0 \text{ on a set of positive measure for every } f,\ \frac{1}{f} \in L^r(\mathbb{R}^n), \tag{9}$$

then $|E(f_0)| \le \liminf |E(g_k)|$.

Proof. Replace m_s by m_0 in Lemmas 4, 5, 6, and 7 to prove that m_0 is of weak type (q, q) on $\mathbb{R}^n$. The rest of the proof requires only minor changes. In particular, in the case when $1/g_k \to 1/f_0$ in L^p, let $A_j(x, s) = A_j(x)$ (note that in Theorem 1, $A_j(x)$ depends on s) and let $A_j(x, 0) = \lim_{s\to 0} A_j(x, s)$. Then

$$A_j(x, 0) \le f_0 \star \mu_j(x) \cdot g_k \star \mu_j(x) \cdot L_j\left(\frac{1}{|f_0 - g_k|}, x\right).$$

Hence $|f_0 \star \mu_j(x) - g_k \star \mu_j(x)| A_j(x, 0) \le c f_0 \star \mu_j(x) \cdot g_k \star \mu_j(x)$ and thus

$$\left|\frac{1}{f_0 \star \mu_j(x)} - \frac{1}{g_k \star \mu_j(x)}\right| \le c A_j(x, 0)^{-1} \le \left[m_0\left(\frac{f_0 g_k}{|g_k - f_0|}\right)(x)\right]^{-1}.$$

The rest of the proof is the same as before. □

We do not know whether m_0 in the hypothesis (9) of Theorem 13 can be replaced by m_∞ as in Theorem 1.

7. There is a variant of Theorem 1 in which the A_s^*-assumption is replaced by a pointwise condition. Let

$$A_s^0(\phi)(x) = \sup_j \phi \star \mu_j(x) \cdot \left(\frac{1}{\phi^s} \star \mu_j(x)\right)^{1/s}.$$

Theorem 14. *Assume that* $0 < p, r, s < \infty$. *If*

$$g_k \to f_0 \text{ in } L^p,$$
$$S = \{x : \sup_k A_s^0(|g_k - f_0|)(x) < \infty\},$$
$$m_\infty f(x) > 0 \text{ on a set of positive measure for every } f, \ \frac{1}{f} \in L^r(\mathbb{R}^n),$$

then $|E(f_0) \cap S| \leq \liminf |E(g_k) \cap S|$.

P r o o f. As before let $E_i = \{x : \limsup f \star \mu_j(x) - \liminf f \star \mu_j(x) > 1/i\}$. If $S_N = \{x \in S : \sup_k A_s^0(|g_k - f_0|)(x) \leq N\}$ and $E_{iN} = E_i \cap S_N$, then

$$|E_{iN}| \leq |\{x \in S_N : M(|g_k - f_0|)(x) > 1/(4i)\}| + |E(g_k) \cap S|.$$

Since for $x \in S_N$,

$$M(|g_k - f_0|)(x) \leq N \left[m_s\left(\frac{1}{|g_k - f_0|}\right)(x)\right]^{-1},$$

we see that by Theorem 8,

$$\begin{aligned} |E_{iN}| &\leq \left|\left\{x \in S_N : m_s\left(\frac{1}{|g_k - f_0|}\right)(x) < 4Ni\right\}\right| + |E(g_k) \cap S| \\ &\leq c(4Ni)^p \|g_k - f_0\|_p^p + |E(g_k) \cap S|. \end{aligned}$$

To complete the proof, let k, N and i go to ∞ in this order. □

There is, of course, a similar *point-wise* version of Theorem 2.

8. We examine now a generalization of the distributional inequality for the minimal operator. Our analysis will be on $\mathbb{R} = \mathbb{R}^1$. Let μ, ν be two positive Borel measures on $\mathbb{R}$, and let $mf(x) = \inf_{x \in I} \frac{1}{|I|} \int_I f$, where I is an interval in $\mathbb{R}$ (see [2], [3]). We consider the following three conditions.

I. There exist constants c_1, c_2 such that $0 < c_1 < 1 < c_2 < \infty$ and for every interval I and $A \subset I$ with $|A| \le c_1|I|$ we have $\mu(I) \le c_2\nu(I \setminus A)$.
II. There exist constants $0 < c < \infty$, $1 < \sigma < \infty$ such that for every $f : \mathbb{R} \to \mathbb{R}^+$, and for every r, $0 < r < \infty$, we have the distributional inequality

$$\mu\{x : mf(x) < r\} \le c\,\nu\{x : f(x) < \sigma r\}.$$

III. There exist constants $0 < c < \infty$, $1 < \sigma < \infty$ such that for every $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and every $f : \mathbb{R} \to \mathbb{R}^+$ we have

$$\int_{\mathbb{R}} \Phi\Big(\frac{1}{mf}\Big)\,d\mu \le c\int_{\mathbb{R}} \Phi\Big(\frac{\sigma}{f}\Big)\,d\nu,$$

where $\Phi(t) = \int_0^t \phi$.

Theorem 15. *The conditions I, II and III are equivalent.*

Proof. I $\Rightarrow$ II. Choose σ so that $2/\sigma \le c_1$, and write the open set $\{x : mf(x) < r\} = \bigcup I_j$, where the I_j are disjoint open intervals. Since $n = 1$, we have for each j, $\int_{I_j} f \le 2r|I_j|$ (see [5]). If $A_j = \{x \in I_j : f(x) \ge \sigma r\}$, then

$$|A_j| \le \frac{1}{\sigma r}\int_{I_j} f \le \frac{2|I_j|r}{\sigma r} = \frac{2|I_j|}{\sigma}.$$

From this we obtain

$$\mu\{x : mf(x) < r\} = \sum_j \mu(I_j) \le c_2\sum_j \nu(I_j \setminus A_j) \le c_2\nu\{x : f(x) < \sigma r\}.$$

II $\Rightarrow$ III. Replace r in II by $1/y$ and multiply by $\phi(y)$ to get

$$\phi(y)\mu\{x : mf(x) < 1/y\} \le c\phi(y)\nu\{x : f(x) < \sigma/y\}.$$

Integrate this inequality in y from 0 to ∞. In the first term let $E = \{(x, y) : mf(x) < 1/y\}$ and note that $\int \chi_E(x, y)\,d\mu(x)\,dx = \mu\{x : mf(x) < 1/y\}$. Substitute this into the integral and interchange the order of integration to obtain the left-hand side of III. The right-hand side is handled in exactly the same way.

III $\Rightarrow$ I. Let $\tau > 2$ and let $c_1 = 1/(\tau\sigma - 1)$. Let I be an interval and $A \subset I$ with $|A| \le c_1|I|$. Define

$$f(x) = \begin{cases} 1/2, & x \in I \setminus A \\ \sigma, & x \in A \\ \infty, & x \notin I. \end{cases}$$

Then
$$\frac{1}{|I|}\int_I f = \frac{1}{|I|}(\sigma|A| + |I\setminus A|/2) \equiv \beta.$$

We observe that $(\sigma - 1/2)|A| = (\beta - 1/2)|I|$. Since $|A| \le c_1|I|$, we get
$$\frac{\beta - 1/2}{\sigma - 1/2} \le c_1 = \frac{1}{\tau\sigma - 1}.$$

From this we see that $\beta \le \frac{\sigma-1/2}{\tau\sigma-1} + \frac{1}{2} \equiv \alpha$ and $\alpha < 1$ since $\sigma - \frac{1}{2} < \frac{1}{2}(\tau\sigma - 1)$. We also observe that $mf(x) \le \alpha$ for $x \in I$ and $mf(x) = \infty$ if $x \notin I$.

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is $\phi(t) = \chi_{[1,1/\alpha]}(t)$, then
$$\Phi(t) = \begin{cases} 0, & 0 \le t \le 1 \\ t-1, & 1 < t \le 1/\alpha \\ 1/\alpha - 1, & t > 1/\alpha. \end{cases}$$

We substitute this into III and get
$$\mu(I)\Phi(1/\alpha) \le c[\nu(A)\Phi(\sigma/\sigma) + \nu(I\setminus A)\Phi(2\sigma)].$$

Since $\Phi(1) = 0$, we obtain $\mu(I)\Phi(1/\alpha) \le c\,\Phi(2\sigma)\nu(I\setminus A)$, and this is condition I. □

Remarks. (i) The measure $\mu = \nu$ satisfies condition I if and only if $\mu\lambda dx$ and $d\mu/dx \in A_\infty$ (see [4]).

(ii) If $(u, v) \in A_p$ and $d\mu = u\,dx$, $d\nu = v\,dx$, then the pair μ, ν satisfies condition I (see Lemma 5 in [1]). The converse, however, is not true. The pair $(\mathrm{e}^{|x|}, \mathrm{e}^{2|x|})$ satisfies I but is not in any A_p.

(iii) A double weight distributional inequality for the maximal operator
$$Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I f$$

has the form $\mu\{x : Mf(x) > y\} \le c\,\nu\{x : f(x) > \sigma y\}$. Unless μ, ν are trivial, this inequality cannot hold. For if $0 < \mu(I), \nu(I) < \infty$ for some interval $I = [a, b]$, then the function $f_N = \chi_{I_N}$, $I_N = [a, a + 1/N]$, satisfies for every N
$$\lim_{y\to 0} \mu\{x : Mf_N(x) > y\} \ge \mu(I), \quad \lim_{y\to 0} \nu\{x : f_N(x) > \sigma y\} = \nu(I_N),$$

and $\nu(I_N) \to 0$ as $N \to \infty$.

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