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HARMONIC AND ENERGY-MINIMAL HOMEOMORPHISMS

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Abstract. These notes were prepared for International School on Nonlinear Analysis, Function Spaces and Applications 9 in Třešť (Czech Republic), September 11–17, 2010. They give an account of some recent developments in which quasiconformal theory and nonlinear elasticity share common problems of compelling mathematical interest. As this interplay developed homeomorphisms with smallest conformal energy became valid and well acknowledged as generalization of conformal mappings in $\mathbb{R}^n$. The main interest lies on two type of mapping problems: i) the existence of homeomorphisms that minimizes the conformal energy; ii) the existence of harmonic homeomorphisms. Here no boundary conditions are imposed. In presenting these topics I will rely on a few recent joint articles with Tadeusz Iwaniec and Leonid V. Kovalev as well as with Kari Astala, Ngin-Tee Koh and Gaven Martin.

1. Setting the stage

Throughout these notes $\mathbb{X}$ and $\mathbb{Y}$ will be nonempty bounded domains in $\mathbb{R}^n$, $n \geq 2$. We will be considering mappings $h : \mathbb{X} \to \mathbb{Y}$ in various Sobolev spaces $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$. The basic concepts are:

1) The differential matrix, also referred to as deformation gradient,

$$Dh(x) = \left[\frac{\partial h^i}{\partial x_j}\right] \in \mathbb{R}^{n\times n}, \quad h = (h^1, \dots, h^n), \tag{1.1}$$

where $\mathbb{R}^{n\times n}$ is the space of $n \times n$-matrices. We reserve the notation $\mathbb{R}^{n\times n}_+$ for the space of matrices with positive determinant.

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2) The Jacobian determinant

$$J(x,h)=\det Dh, \quad J(x,h)\,\mathrm{d}x=\mathrm{d}h^1\wedge\cdots\wedge\mathrm{d}h^n. \tag{1.2}$$

More generally, to all pairs of ordered ℓ-tuples $I=(i_1,\dots,i_\ell)$ and $J=(j_1,\dots,j_\ell)$, where $1\le i_1<\cdots<i_\ell\le n$ and $1\le j_1<\cdots<j_\ell\le n$, there correspond

3) the $(\ell\times\ell)$-subdeterminants and their matrices

$$\frac{\partial h^I}{\partial x_J}=\frac{\partial\left(h^{i_1},\dots,h^{i_\ell}\right)}{\partial\left(x_{j_1},\dots,x_{j_\ell}\right)}, \quad D^{\ell\times\ell}h=\left[\frac{\partial h^I}{\partial x_J}\right]\in\mathbb{R}^{\binom{n}{\ell}\times\binom{n}{\ell}}. \tag{1.3}$$

For $\ell=n-1$, we obtain

4) Cramer's cofactor matrix

$$D^\sharp h=\left[(-1)^{i+j}\frac{\partial\left(h^1,\dots,h^{i-1},h^{i+1},\dots,h^n\right)}{\partial\left(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n\right)}\right]\in\mathbb{R}^{n\times n}. \tag{1.4}$$

By convention, $\frac{\partial h^I}{\partial x_J}=1$ if $\ell=0$. In this way the total number of all subdeterminants is

$$\sum_{\ell=0}^{n}\binom{n}{\ell}^2=\binom{2n}{n}. \tag{1.5}$$

The $\ell\times\ell$-minors govern the infinitesimal deformations of ℓ-dimensional objects.

When $n=2$, we write $h=u+\mathrm{i}v\colon\mathbb{X}\to\mathbb{Y}$, where $\mathbb{X},\mathbb{Y}\subset\mathbb{C}=\{z=x+\mathrm{i}y: x,y\in\mathbb{R}\}$. The complex derivatives of h take the form

$$h_z=\frac{\partial h}{\partial z}=\frac{1}{2}\left(\frac{\partial h}{\partial x}-\mathrm{i}\frac{\partial h}{\partial y}\right) \quad\text{and}\quad h_{\bar z}=\frac{\partial h}{\partial z}=\frac{1}{2}\left(\frac{\partial h}{\partial x}+\mathrm{i}\frac{\partial h}{\partial y}\right). \tag{1.6}$$

1.1. *n*-harmonic hyperelasticity. The general law of hyperelasticity tells us that there exists an energy integral

$$\mathcal{E}[h]=\int_{\mathbb{X}}E(x,h,Dh)\,\mathrm{d}x, \tag{1.7}$$

where $E\colon\mathbb{X}\times\mathbb{Y}\times\mathbb{R}^{n\times n}\to\mathbb{R}$ is a given *stored-energy* function characterizing mechanical properties of the material. The mathematical models of nonlinear elasticity have been pioneered by Antman [2], Ball [7] and Ciarlet [12]. In these lectures we are mainly interested in the *n-harmonic energy*,

$$\mathcal{E}[h]=\int_{\mathbb{X}}|Dh(x)|^n\,\mathrm{d}x. \tag{1.8}$$

Here and in what follows we conveniently work with the Hilbert-Schmidt norm of a linear map A, defined by the rule $|A|^2 = \operatorname{Tr}(A^t A)$. The primary question is the existence of homeomorhisms that minimizes the n-harmonic energy integral. We impose no boundary conditions on homeomorphisms $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. Such a homeomorphism, if exists, is called *energy-minimal homeomorphism.* The existence of globally invertible energy-minimal mapping is one of the primary pursuits in the mathematical models of nonlinear elasticity [8], [52].

In another direction, we recall Geometric Function Theory in $\mathbb{R}^n$ and its governing variational integrals. A mapping $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is conformal at $x \in \mathbb{X}$ if $|Dh(x)|^n = n^{n/2} J(x,h)$. This can be expressed in the form of a nonlinear Cauchy-Riemann system of PDEs:

$$D^*h(x) \circ Dh(x) = J(x,h)^{\frac{2}{n}} \mathbf{I}. \tag{1.9}$$

It is evident that the n-harmonic energy of all conformal deformations $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is the same. Indeed, we have

$$\mathcal{E}[h] = \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x = n^{\frac{n}{2}} \int_{\mathbb{X}} J(x,h)\,\mathrm{d}x = n^{\frac{n}{2}} |\mathbb{Y}|. \tag{1.10}$$

For other homeomorphisms $g\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathcal{W}^{1,n}(\mathbb{X},\mathbb{Y})$, we have only the lower bound due to Hadamard's inequality:

$$E[g] \geq \int_{\mathbb{X}} |Dg|^n \geq n^{\frac{n}{2}} \int_{\mathbb{X}} J(x,g)\,\mathrm{d}x = n^{\frac{n}{2}} |\mathbb{Y}|.$$

Thus, conformal mappings $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ would be an obvious choice for the minimizer of (1.8). Even in the plane, multiply connected domains are of various conformal type. The existence of an energy-minimal homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of Dirichlet energy may be interpreted as saying that the Cauchy-Riemann equation $\bar\partial h = 0$ admits a diffeomorphic solution in the least squares sense, meaning that $\|\bar\partial h\|_{\mathcal{L}^2}$ assumes its minimum. For this reason, energy-minimal homeomorphisms are known under the name *least squares conformal mappings* in the computer graphics literature [40], [47].

It is rare in higher dimensions that two topologically equivalent domains are conformally equivalent, because of Liouville's rigidity theorem. From this point of view quasiconformal theory [4], [21], [26], [33] offers significantly larger class of mappings. The most general class of mappings for which we are hoping to build the theory is the following.

Definition 1.1. A weakly differentiable mapping $f\colon \mathbb{Y} \to \mathbb{X}$ has *finite distortion* if

- The Jacobian determinant of f is locally integrable.
- There is a measurable function $K_O(y) \geq 1$, finite a.e. such that

$$|Df(y)|^n \leq K_O(y) J(y,f) \quad \text{a.e.} \tag{1.11}$$

The distortion inequality (1.11) merely asks that the differential $Df(y)$ vanishes at those points y where the Jacobian $\det Df(y) = J(y,f) = 0$. This seems to be a minimal requirement for a mapping to carry any geometric information. The smallest function $K_O(y) = K_O(y,f) \geq 1$ for which the distortion inequality (1.11) holds is called the *outer distortion* of f,

$$K_O(x,f) = \begin{cases} \frac{|Df(x)|^n}{n^{n/2}\, J(x,f)} & \text{if } J(x,f) \in \mathbb{R}^{n\times n}_+, \\ 1 & \text{if } J(x,f) = 0. \end{cases} \tag{1.12}$$

Suppose f has finite distortion. Then the *inner distortion* of f is defined by the rule

$$K_I(x,f) = \begin{cases} \frac{|D^\sharp f(x)|^n}{n^{n/2} J(x,f)^{n-1}} & \text{if } J(x,f) \in \mathbb{R}^{n\times n}_+, \\ 1 & \text{if } J(x,f) = 0. \end{cases} \tag{1.13}$$

These two distortions are borderline cases of

$$K_\ell(x,f) = \begin{cases} \frac{|D^{\ell\times\ell} f(x)|^n}{\binom{n}{\ell}^{n/2} J(x,f)^\ell} & \text{if } J(x,f) \in \mathbb{R}^{n\times n}_+, \\ 1 & \text{if } J(x,f) = 0. \end{cases} \tag{1.14}$$

We obtain mappings of bounded distortion, also called quasiregular mappings or quasiconformal mappings if f is a homeomorphism, when $K_O \in \mathscr{L}^\infty(\mathbb{Y})$. The theory of mappings of bounded distortion is by now well understood, see the monographs [48] by Reshetnyak, [49] by Rickman, [33] by Iwaniec and Martin, and [4] by Astala, Iwaniec and Martin. Recently, systematic studies of mappings of finite distortion have emerged in Geometric Function Theory (GFT) [33].

One of the most appealing recent discoveries in the GFT is a connection between the conformal energy of a homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ and the inner distortion function of the inverse mapping $f\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ ([13], [23], [24]).

Theorem 1.2. *Let* $f\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ *be a homeomorphism of the Sobolev class* $\mathcal{W}^{1,n-1}_{\mathrm{loc}}(\mathbb{Y},\mathbb{X})$. *Assume that* $K_I \in \mathcal{L}^1(\mathbb{Y})$. *Then the inverse map* $h(x) = f^{-1}(x)$ *belongs to the Sobolev class* $\mathcal{W}^{1,n}(\mathbb{X},\mathbb{Y})$ *and we have*

$$n^{\frac{n}{2}} \int_{\mathbb{Y}} K_I(y,f)\,\mathrm{d}y = \int_{\mathbb{X}} |Dh(x)|^n\,\mathrm{d}x. \tag{1.15}$$

One should note that in this transition the polyconvex variational integral on the left-hand side becomes convex on the right-hand side, a rarity that we shall enjoy when studying extremal quasiconformal mappings. The classical Teichmüller theory is concerned, broadly speaking, with extremal mappings between Riemann surfaces. The extremal Teichmüller mapping is exactly the one whose distortion function has the smallest possible supremum norm. The existence and uniqueness of such an extremal quasiconformal map within a given homotopy class is the heart of Teichmüller's theory. Now, in view of the identity (1.15), minimizing the $\mathcal{L}^1$-norm of the inner distortion amounts to the study of *n-harmonic energy* of the inverse mapping $h = f^{-1}\colon \mathbb{X} \to \mathbb{Y}$,

$$\mathcal{E}[h] = \int_{\mathbb{X}} |Dh(x)|^n\,\mathrm{d}x. \tag{1.16}$$

2. Energy-minimal homeomorphisms: existence and nonexistence

Annuli are where one first observes nontrivial conformal invariants.

2.1. Annuli and *n*-harmonics ([3], [37]). Consider mappings $h\colon \mathbb{A} \to \mathbb{A}^*$ between concentric spherical rings in $\mathbb{R}^n$, also called annuli.

$$\mathbb{A} = \mathbb{A}(r,R) = \{x \in \mathbb{R}^n : r < |x| < R\}, \quad 0 \le r < R < \infty,$$
$$\mathbb{A}^* = \mathbb{A}(r_*,R_*) = \{y \in \mathbb{R}^n : r_* < |y| < R_*\}, \quad 0 \le r_* < R_* < \infty.$$

These domains are of different conformal type unless the ratio of the two radii is the same for both annuli.

Theorem 2.1 ([50]). *Let* $\mathbb{A}$ *and* $\mathbb{A}^*$ *be two planar annuli. There exists a conformal homeomorphism* $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ *if and only if the annuli have the same modulus; that is,*

$$\operatorname{Mod}\mathbb{A} := \log\frac{R}{r} = \log\frac{R_*}{r_*} =: \operatorname{Mod}\mathbb{A}^*.$$

Moreover, up to the rotation of the annuli, every such map takes the form

$$h(z) = \begin{cases} \dfrac{r_* z}{r}, & \textit{if preserving the order of boundary components,} \\ \dfrac{r R_*}{z}, & \textit{if reversing the order of boundary components.} \end{cases}$$

Concerning the domains of higher connectivity in dimension $n = 2$, the conformal type of a domain of connectivity $\ell > 2$ is determined by $3\ell - 6$ parameters, called Riemann moduli of the domain.[1] For our purposes we may as well restrict our attention to the sense-preserving homeomorphisms that also preserve the order of boundary components (inner and outer).

Now suppose that the domain annulus is substantially thicker than $\mathbb{A}^* \subset \mathbb{R}^n$. Our computation reveals, despite physical intuition, that in order to minimize the energy (1.8) of a map $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ we must "hammer" a part of $\mathbb{A} = A(r, R)$ into a circle. An effect of this hammering procedure is that the minimizer will no longer satisfy the n-harmonic equation

$$\operatorname{div}\big(|Dh|^{n-2} Dh\big) = 0, \quad h \in \mathcal{W}^{1,n}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n). \tag{2.1}$$

For simplicity, we will demonstrate this situation only in the planar case, and refer to [37] for the complete picture of the similar phenomena in all dimensions. In dimension $n = 2$ the minimizer is a harmonic homeomorphism (no hammering) if and only if the *Nitsche bound*

$$\frac{R_*}{r_*} \geq \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right) \quad \text{i.e.} \quad \operatorname{Mod} \mathbb{A}^* \geq \log \cosh \operatorname{Mod} \mathbb{A} \tag{2.2}$$

holds. If this condition fails we then have a sub-annulus $A(\rho, R) \subset A(r, R) = \mathbb{A}$ which together with $\mathbb{A}^*$ forms a critical Nitsche configuration. Precisely, this sub-annulus is determined by the condition $\frac{R_*}{r_*} = \frac{1}{2}\left(\frac{R}{\rho} + \frac{\rho}{R}\right)$. By way of illustration, the extremal map we are referring to takes the form

$$h(z) = \begin{cases} \frac{z}{|z|} r_* & r < |z| \leq \rho, \quad \text{hammering part,} \\ \frac{1}{2}\left(\frac{z}{\rho} + \frac{\rho}{\bar{z}}\right) r_* & \rho \leq |z| \leq R, \quad \text{Nitsche map.} \end{cases} \tag{2.3}$$

It is true though somewhat less obvious, that this mapping h is a $\mathcal{W}^{1,2}$-limit of homemorphisms from $\mathbb{A}$ onto $\mathbb{A}^*$ and its energy is smaller than that of

[1] In this context the mappings are orientation preserving.

any homeomorphism from $\mathbb{A}$ onto $\mathbb{A}^*$ ([3], [37]). Actually, we proved that there is the similar phenomena in all dimensions. In [37] we found precise condition of the form

$$\operatorname{Mod}\mathbb{A}^* < \Phi(\operatorname{Mod}\mathbb{A}), \quad \text{where as always } \operatorname{Mod}\mathbb{A} = \log\frac{R}{r}, \tag{2.4}$$

under which the hammering of the thick annulus $\mathbb{A}$ is necessary in order to minimize the conformal energy (1.8).

Of course, studying the extremal deformations between circular annuli it is natural to look for the radially symmetric solutions of the n-harmonic equation (2.1), where $h(x) = H(|x|)\frac{x}{|x|}$. However, the answer to the question of whether these maps minimize the n-harmonic energy (1.8) is not obvious. When $n = 2$ or $n = 3$, the answer is "yes", [37].

Surprisingly, for $n \geq 4$, the answer depends on the conformal width of $\mathbb{A}^*$.

Theorem 2.2 ([37])**.** *For each $n \geq 4$, there are annuli $\mathbb{A}, \mathbb{A}^* \subset \mathbb{R}^n$ such that the n-harmonic energy does not assume its minimum value on any radial mapping.*

2.2. Dirichlet energy ([28])**.** Here we establish the existence of homeomorphisms between two bounded planar domains $\mathbb{X}$ and $\mathbb{Y}$ that minimize the Dirichlet energy,

$$\mathcal{E}[f] = \int_{\mathbb{X}} |Dh|^2 = 2\int_{\mathbb{X}} (|h_z|^2 + |h_{\bar{z}}|^2) \tag{2.5}$$

among all $\mathcal{W}^{1,2}$-homeomorphisms $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. As we noticed in §2.1, such an energy-minimal homeomorphism may fail to exist when a minimizing sequence collapses, at least partially, onto the boundary of $\mathbb{Y}$.

In general, minimizing the energy among homeomorphisms need not lead to the Laplace equation, see e.g. (2.3). Harmonicity is lost exactly at the branch points. Outside the branch set the extremal mappings are indeed harmonic. This latter fact follows from Radó-Kneser-Choquet Theorem [16, p. 29] which allows us to apply the Poisson modification (and thus decrease the energy of a non-harmonic mapping) without losing injectivity. However, the existence of a harmonic homeomorphism does not imply the existence of an energy-minimal one.

As we have already pointed out, energy-minimal homeomorphisms for simply connected domains are obtained from the Riemann mapping theorem. The doubly connected case, being next in the order of complexity, is the subject of our next result.

Theorem 2.3 ([28]). *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are bounded doubly connected domains in $\mathbb{C}$ such that $\operatorname{Mod}\mathbb{X} \le \operatorname{Mod}\mathbb{Y}$. Then there exists an energy-minimal homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, which is unique up to a conformal change of variables in $\mathbb{X}$.*

Hereafter $\operatorname{Mod}\mathbb{X}$ stands for the *conformal modulus* of $\mathbb{X}$. Any bounded doubly connected domain $\mathbb{X} \subset \mathbb{C}$ is conformally equivalent to some circular annulus $\{z : r < |z| < R\}$ with $0 \le r < R < \infty$. The ratio R/r, being independent of the choice of conformal equivalence (Theorem 2.1), defines $\operatorname{Mod}\mathbb{X} := \log R/r$. It is rather surprising that the existence result for energy-minimal homeomorphisms relies only on the conformal modulus of the target. Indeed, the energy minimization problem is invariant only with respect to a conformal change of variable in the domain, not in the target.

In the converse direction we showed that there exists no energy-minimal diffeomorphism when $\operatorname{Mod}\mathbb{Y} \le \Phi(\operatorname{Mod}\mathbb{X})$. Here $\Phi\colon (0,\infty) \to (0,\infty)$ is a certain function asymptotically equal to the identity at infinity, $\lim_{t\to\infty} \Phi(t)/t = 1$. It is in this asymptotic sense that Theorem 2.3 is sharp. Precisely, we proved the following nonexistence results.

Theorem 2.4. ([28]). *There is a nondecreasing function $\Upsilon\colon (0,\infty) \to (0,1)$ such that $\lim_{\tau\to\infty} \Upsilon(\tau) = 1$ and the following holds. Whenever two bounded doubly connected domains $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{C}$ admit an energy-minimal homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, we have*

$$\operatorname{Mod}\mathbb{Y} \ge (\operatorname{Mod}\mathbb{X}) \cdot \Upsilon(\operatorname{Mod}\mathbb{X}). \tag{2.6}$$

Specifically, one can take

$$\begin{gathered} \Upsilon(\tau) = \exp\left(-\frac{\pi^2}{2\tau}\right) \cdot \Lambda\left(\coth\frac{\pi^2}{2\tau}\right), \\ \textit{where} \quad \Lambda(t) = \frac{\log t - \log(1+\log t)}{2+\log t}, \quad t \ge 1. \end{gathered} \tag{2.7}$$

We conjecture that (2.6) can be specified as in the circular annuli case (2.2).

Conjecture 2.5 ([28]). *If two bounded doubly connected domains $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{C}$ admit an energy-minimal homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, then*

$$\operatorname{Mod}\mathbb{Y} \ge \log\cosh\operatorname{Mod}\mathbb{X}.$$

Moreover, if both sides are finite and equal, then $\mathbb{Y}$ is a circular annulus.

The crucial idea of the proof of Theorem 2.3 was to consider a one-parameter family of variational problems in which $\mathbb{X}$ changes continuously while $\mathbb{Y}$ remains fixed. We established strict monotonicity of the minimal energy as a function of the conformal modulus of $\mathbb{X}$. We denote the set of all sense-preserving $\mathscr{W}^{1,2}$-homeomorphisms $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by $\mathsf{H}^{1,2}(\mathbb{X},\mathbb{Y})$. When $\mathsf{H}^{1,2}(\mathbb{X},\mathbb{Y})$ is nonempty, we define

$$\mathsf{E}_{\mathsf{H}}(\mathbb{X},\mathbb{Y}) = \inf\{\mathcal{E}[h] : h \in \mathsf{H}^{1,2}(\mathbb{X},\mathbb{Y})\}. \tag{2.8}$$

By virtue of the density of diffeomorphisms in $\mathsf{H}^{1,2}(\mathbb{X},\mathbb{Y})$, see §4.2, the minimization of energy among sense-preserving diffeomorphisms leads to the same value $\mathsf{E}_{\mathsf{H}}(\mathbb{X},\mathbb{Y})$. Let us emphasize that bounded subsets of $\mathsf{H}^{1,2}(\mathbb{X},\mathbb{Y})$ are lacking compactness, due to the loss of injectivity in passing to a limit of homeomorphisms. That is why we introduced the class of so-called *deformations*. Precisely:

Definition 2.6. A mapping $h\colon \mathbb{X} \to \overline{\mathbb{Y}}$ is called a *deformation* if

- $h \in \mathscr{W}^{1,2}(\mathbb{X},\mathbb{Y})$;
- the Jacobian $J_h := \det Dh$ is nonnegative a.e. in $\mathbb{X}$;
- $\int_{\mathbb{X}} J_h \le |\mathbb{Y}|$;
- there exist sense-preserving homeomorphisms $h_j\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ such that
  (a) $h_j \to h$ uniformly on compact subsets of $\mathbb{X}$ and
  (b) $\operatorname{dist}(h_j, \partial\mathbb{Y}) \to \operatorname{dist}(h, \partial\mathbb{Y})$ uniformly on $\overline{\mathbb{X}}$.

The set of deformations $h\colon \mathbb{X} \to \overline{\mathbb{Y}}$ is denoted by $\mathcal{D}(\mathbb{X},\mathbb{Y})$.

This class contains $\mathsf{H}^{1,2}(\mathbb{X},\mathbb{Y})$ and is closed under weak limits in $\mathscr{W}^{1,2}(\mathbb{X},\mathbb{Y})$. Due to the weak lower semicontinuity of the Dirichlet energy, there exists $h \in \mathcal{D}(\mathbb{X},\mathbb{Y})$, called an energy-minimal deformation, such that

$$\mathcal{E}[h] = \mathsf{E}(\mathbb{X},\mathbb{Y}) := \inf\{\mathcal{E}[h] : h \in \mathcal{D}(\mathbb{X},\mathbb{Y})\}.$$

Furthermore, the class of deformations is closed under compositions with self-diffeomorphisms of $\mathbb{X}$; we can perform inner variation, which yield that the Hopf differential (§4.1) of an energy-minimal deformation is holomorphic in $\mathbb{X}$ and real on its boundary. We gain additional information about the Hopf differential from the Reich-Walczak-type inequalities which is where the conformal moduli of $\mathbb{X}$ and $\mathbb{Y}$ enter the stage.

2.3. Total conformal energy ([34])**.** As we noticed in §2.1, passing to a minimizing sequence $h_j\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with homeomorphisms of bounded conformal energy may result in a non-injective mapping. Actually, if $\mathbb{X}$ has at least two boundary components, then the only way that one can loose injectivitity is the hammering process; that is, a minimizing sequence collapses, at least partially, onto the boundary of $\mathbb{Y}$. Precisely, we have the following result.

Theorem 2.7 ([35]). *Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ be bounded domains of the same topological type having k boundary components, $k = 2, 3, \dots$. Suppose there is a sequence of homeomorphisms $h_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ converging weakly in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ to h. Then h is continuous and differentiable almost everywhere, $\mathbb{Y} \subset h(\mathbb{X}) \subset \overline{\mathbb{Y}}$. It has a right inverse $f \colon \mathbb{Y} \to \mathbb{X}$ of bounded variation; that is,*

$$h \circ f = \mathrm{id} \colon \mathbb{Y} \to \mathbb{Y}.$$

Next, we want to study energy functionals which allow us to deform the body back, again with finite energy. In view of examples of the extremal harmonic mappings for the Dirichlet energy (the Nitsche map) it becomes clear that we must ensure finite conformal energy not only for the mapping in question but also for its inverse. Our model example, from which other *conformally coercive functionals* are derived, is the average of two n-harmonic integrals:

$$\mathcal{E} = \mathcal{E}[h, f] = \frac{\alpha}{n^{\frac{n}{2}} |\mathbb{Y}|} \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x + \frac{\beta}{n^{\frac{n}{2}} |\mathbb{X}|} \int_{\mathbb{Y}} |Df(y)|^n \, \mathrm{d}y, \tag{2.9}$$

where $\alpha, \beta > 0$, $\alpha + \beta = 1$ and $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is an orientation preserving homeomorphism (perchance in a given homotopy class) and $f \colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ stands for its inverse. With the aid of the identity (1.15) we are reduced to a variational integral for h alone,

$$\mathcal{E}[h] = \int_{\mathbb{X}} \big[a |Dh(x)|^n + b K_I(x, h) \big] \, \mathrm{d}x, \quad \text{where } a, b > 0.$$

The key is that this integral is well defined and polyconvex in a class of non-injective mappings of integrable distortion. The direct method of the calculus of variations comes in handy for establishing the existence of minimizers. However, it is not clear why a minimizer should be an injective map of $\mathbb{X}$ onto $\mathbb{Y}$. We always assume that $\mathbb{X}$ and $\mathbb{Y}$ admit at least one homeomorphism of finite *total n-harmonic energy* defined by (2.10).

Theorem 2.8 ([34]). *Let $\mathbb{X}$ and $\mathbb{Y}$ be domains in $\mathbb{R}^n$ with at least two but finitely many boundary components. Then the total n-harmonic energy $\mathcal{E}[h]$ assumes its minimum value among all homeomorphisms $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.*

In view of the identity (1.15) the energy (2.9) translates into the *mean total distortion*,

$$\mathcal{E} = \alpha \fint_{\mathbb{Y}} K_I(y, f) \, \mathrm{d}y + \beta \fint_{\mathbb{X}} K_I(x, h) \, \mathrm{d}x.$$

Clearly, the conformal deformations are exactly the ones having the absolute minimum energy; that is, $\mathcal{E} = 1$. However, it is rare in higher dimensions that two topologically equivalent domains are conformally equivalent.

Theorem 2.9 ([34]). *Among all deformations $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of planar annuli the smallest mean distortion is attained on a $\mathcal{C}^\infty$-smooth radial map. This radial map is a unique minimizer up to conformal authomorphisms of the annuli.*

This raises a very interesting question: Are the mappings of smallest mean distortion always $\mathcal{C}^\infty$-smooth? As a starting point one can ask: Does a deformation minimizes the energy $\mathcal{E}$ exactly when it satisfies the corresponding Lagrange-Euler equation¡ Certainly, this is not the case for general variational integrals such as the n-harmonic energy. Such a discrepancy occurs exactly at the points where the minimizer fails to be injective. An unexpected lack of symmetry speaks convincingly of the complexity of this problem:

Theorem 2.10 ([34]). *For each $n \geq 4$ there are annuli $\mathbb{A}, \mathbb{A}^* \subset \mathbb{R}^n$ such that the minimum of the total n-harmonic energy is attained, but not on a radial mapping.*

Concerning the minimization problems between annuli, we introduced and heavily relied on the concept of *free Lagrangians* [37]. In 1977 a novel approach towards minimization of polyconvex energy functionals was developed and published by Ball [7]. The underlying idea was to view the integrand as convex function of null Lagrangians. The term null Lagrangian pertains to a nonlinear differential expression whose integral over any open region depends only on the boundary values of the map, see [11], [17], [27]. But we are concerned with homeomorphisms $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ which are not prescribed on the boundary. There still exist some nonlinear differential forms, called free Lagrangians, defined on a given homotopy class of homeomorphisms, whose integral means remain independent of the deformation. These are rather special null Lagrangians, see [37], [34]. In §5 we will give an elementary proof for Schottky's theorem employing free-Lagrangians [5].

3. Harmonic mappings, the Nitsche conjecture ([29], [30])

Harmonic mappings are only stationary solutions of the Dirichlet integral; they do not always minimize the energy. Thus the hammering result (2.3) does not rule out the existence of univalent harmonic mappings from $\mathbb{A}$ onto $\mathbb{A}^*$; known as the Nitsche conjecture (1962).

The Nitsche Conjecture ([41]). *A harmonic homeomorphism $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ exists if and only if*

$$\frac{R_*}{r_*} \geq \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right). \tag{3.1}$$

NITSCHE studied the existence of doubly connected minimal surfaces $\mathbb{S} \subset \mathbb{R}^3$ with prescribed boundary curves [43]. Minimal surfaces in $\mathbb{R}^3$ are fundamental forms in mathematics and physics [14], [15], [45]. For example, the minimal surface joining some pairs of coaxial circles in parallel planes has the shape of a catenoid, a configuration that is extremal for numerous problems [7], [25], [39], [42], [43], [44], [46]. The basic result is that no such surface exists if the curves are too far from each other relative to their diameter. NITSCHE considered curves lying in parallel planes. One way of measuring farness is to look at the conformal modulus of the surface. It follows from Theorem 3.1 that a slab of one-sided catenoid has also the largest conformal modulus among minimal graphs over a given annulus. Indeed, a doubly connected minimal graph $\mathbb{S} \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ induces a harmonic homeomorphism $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{Y}$, where $\mathbb{Y} \subset \mathbb{C}$ is the doubly connected domain over which the graph lies. NITSCHE [41] discovered that the existence of a harmonic homeomorphism $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{Y}$ implies an upper bound on the conformal modulus of $\mathbb{S}$ in terms of $\operatorname{Mod} \mathbb{A}$. When the configuration domain $\mathbb{Y} = \mathbb{A}^* = A(r_*, R_*)$ is a circular annulus, he conjectured that necessary and sufficient condition for such a mapping to exist is (3.1).

Theorem 3.1 ([30]). *The Nitsche conjecture holds.*

It should be noted that harmonicity of a function $h = h(z)$ is invariant under conformal change of the z-variable. Therefore, the Nitsche bound remains valid for harmonic homeomorphism defined on any doubly connected domain whose conformal modulus coincides with that of $\mathbb{A}$. It is therefore of interest to look at the role of the boundary curves in the target annulus as well. The circular shape of the outer boundary turns out to be inessential, there remains a substitute of the Nitsche bound in terms of the integral means over the circles $\mathbb{T}_\rho = \{z \in \mathbb{C} : |z| = \rho\}$, $r < \rho < R$. In our generalized form of the Nitsche bound the target is a half circular annulus; that is, a doubly connected domain $\mathcal{A}^*$ whose inner boundary is a circle $\mathbb{T}_{r_*}$. We do not specify the outer boundary of $\mathcal{A}^*$ as it can be arbitrary. Let $\mathcal{H}(\mathbb{A}, \mathcal{A}^*)$ denote the class of orientation preserving harmonic homeomorphisms $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathcal{A}^*$ which take the inner boundary of $\mathbb{A}$ into the inner boundary of $\mathcal{A}^*$.

Theorem 3.2 (GENERALIZED NITSCHE BOUND [30]). *For every $h \in \mathcal{H}(\mathbb{A}, \mathcal{A}^*)$, we have*

$$\left[\fint_{\mathbb{T}_\sigma} |h|^2 \right]^{\frac{1}{2}} \geq \frac{1}{2}\left(\frac{\sigma}{r} + \frac{r}{\sigma}\right) r_*, \quad r < \sigma < R. \tag{3.2}$$

If equality occurs at some radius $\sigma \in (r, R)$, then it holds for every $\sigma \in (r, R)$. In this case h takes the form $h(z) = \frac{1}{2}\big(\frac{z}{r} + \frac{r}{\bar z}\big) r_* e^{i\alpha}$.

Such a more general statement not only strengthens the Nitsche Conjecture but also was the key to the proof. In fact Theorem 3.2 should be viewed as a sharp lower estimate for the growth of integral means of harmonic mappings under certain initial constraints. These constraints concern topological behavior of h near the inner boundary $\mathbb{T}_r$ rather than its boundary values. Now, to make (3.2) valid for such a harmonic function $h\colon A(r,R) \to \mathbb{C}$, we only need the following three initial conditions:

(I) $h\colon \mathbb{T}_r \to \mathbb{T}_{r_*}$ is a homeomorphism of winding number 1;

(II) $\dfrac{\mathrm{d}}{\mathrm{d}\sigma} \fint_{\mathbb{T}_\sigma} |h|^2 \geq 0$, at $\sigma = r$;

(III) $\displaystyle\int_{\mathbb{T}_r} \det Dh \geq 0$.

In §5 we give a proof of Schottky's theorem employing circular means of $|h|^2$ and condition (I). This proof is from [30].

A natural question arises if we consider minimal surfaces instead of minimal graphs. Here one might expect the extremal surface to be a symmetric slab of catenoid for which

$$\frac{R_*}{r_*} \geq \frac{1}{2}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right). \tag{3.3}$$

We must assume that the mapping is not nullhomotopic within the punctured plane $\mathbb{C} \setminus \{0\}$.

Conjecture 3.3 ([30])**.** *Let $h\colon A(r,R) \to A(r_*,R_*)$ be a harmonic mapping (not necessarily injective or surjective) with nonzero winding number; that is,*

$$\int_{\mathbb{T}_\sigma} \frac{\mathrm{d}h}{h} \neq 0, \quad \textit{for some (equivalently, for all) } \sigma \in (r,R).$$

Then

$$\frac{R_*}{r_*} \geq \frac{1}{2}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right).$$

Equality occurs for the double cover Nitsche map:

$$h(z) = \frac{1}{2}\left(\frac{z}{\sqrt{rR}} + \frac{\sqrt{rR}}{\bar{z}}\right) \quad \textit{for } r < |z| < R.$$

4. Hopf differentials and diffeomorphic approximation of Sobolev homeomorphisms

4.1. Hopf differentials ([31]). A quadratic differential on a domain $\mathbb{X}$ in the complex plane $\mathbb{C}$ takes the form $Q = F(z)\,\mathrm{d}z \otimes \mathrm{d}z$, where F is a complex function on $\mathbb{X}$. For a complex harmonic function $h\colon \mathbb{X} \to \mathbb{C}$, the associated Hopf differential

$$Q_h = h_z \overline{h_{\bar z}}\,\mathrm{d}z \otimes \mathrm{d}z$$

is holomorphic, meaning that

$$\frac{\partial}{\partial \bar z}(h_z \overline{h_{\bar z}}) = 0. \tag{4.1}$$

Naturally, the Sobolev space $\mathcal{W}^{1,2}_{\mathrm{loc}}(\mathbb{X},\mathbb{C})$ should be considered as the domain of definition of equation (4.1). This places $h_z\overline{h_{\bar z}}$ in $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{X})$, so the complex Cauchy-Riemann derivative $\frac{\partial}{\partial \bar z}$ applies in the sense of distribution. By Weyl's lemma $h_z\overline{h_{\bar z}}$ is a holomorphic function.

Every energy-minimal deformation $h \in \mathcal{D}(\mathbb{X},\mathbb{Y})$, §2.2, is stationary; that is,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{E}[h \circ \phi_t^{-1}] = 0 \tag{4.2}$$

for every family of diffeomorphisms $\phi_t\colon \Omega \to \Omega$ which depend smoothly on the parameter $t \in \mathbb{R}$ and satisfy $\phi_0 = id$.

It follows from the fundamental theorem of Radó, Kneser and Choquet that for mappings between domain $\mathbb{X},\mathbb{Y}$ in $\mathbb{C}$ any minimizer of the Dirichlet energy is harmonic outside the branch set and therefore has holomorphic Hopf differential. A natural question arises whether a Sobolev homeomorphism $h \in \mathcal{W}^{1,2}_{\mathrm{loc}}(\mathbb{X},\mathbb{C})$ with holomorphic Hopf differential is harmonic. This question was originated in a series of papers by Eells, Lemaire and Sealey [19], [20], [51].

Theorem 4.1 ([31]). *Every homeomorphism h of Sobolev class $\mathcal{W}^{1,2}_{\mathrm{loc}}(\mathbb{X},\mathbb{C})$ that satisfies equation* (4.1) *is harmonic.*

Especially, if a Hopf differential $Q_h = h_z\overline{h_{\bar z}}\,\mathrm{d}z^2$ is holomorphic for some $\mathcal{C}^1$-mapping h, then h is harmonic at the points, where the Jacobian determinant $J(z,h) := \det Dh = |h_z|^2 - |h_{\bar z}|^2 \neq 0$, see also [18, 10.5]. Here the assumption that $J(z,h) \neq 0$ is critical, see e.g. (2.3).

The Eells-Lemaire problem under the additional assumption that h is a quasiconformal homeomorphism was settled earlier by Hélein [22] in the affirmative. Theorem 4.1 dispose with the quasiconformality condition and treat general planar $\mathcal{W}^{1,2}$-homeomorphisms. Since the inverse of such a homeomorphism need not be in any Sobolev class, some difficulties were expected. They were overcome with the aid of our approximation theorem which we will present next.

4.2. Diffeomorphic approximation of Sobolev homeomorphisms ([32]). By definition, the Sobolev space $\mathcal{W}^{1,p}(\mathbb{X},\mathbb{R})$, $1 \le p < \infty$, is the completion of $\mathcal{C}^\infty$-smooth real functions having finite Sobolev norm

$$\|u\|_{\mathcal{W}^{1,p}(\mathbb{X})} = \|u\|_{\mathcal{L}^p(\mathbb{X})} + \|\nabla u\|_{\mathcal{L}^p(\mathbb{X})} < \infty.$$

Question 1. Let $n = 2, 3$. Suppose that $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a homeomorphism in $\mathcal{W}^{1,p}(\mathbb{X},\mathbb{R}^n)$. Can h be approximated by diffeomorphisms $h_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathcal{W}^{1,p}(\mathbb{X},\mathbb{R}^n)$?

A different, but equivalent, version of Question 1 asks for h_j to be piecewise affine invertible mappings. In this form the approximation problem was put forward by J. M. Ball [9], [10], who attributed it to L. C. Evans. The paper [32] provides an affirmative solution of the Ball-Evans problem in the planar case when $1 < p < \infty$. The case $p = 1$ remains open.

Our construction of an approximating diffeomorphism heavily relies on the following p-harmonic replacement argument. Let $\mathbb{U} \subset \mathbb{C}$ be a bounded simply connected domain. For any $h_\circ \in \mathcal{W}^{1,p}(\mathbb{U},\mathbb{C}) \cap \mathcal{C}(\overline{\mathbb{U}})$, $1 < p < \infty$, there exists a unique *coordinate-wise p-harmonic mapping* $h \colon \mathbb{U} \to \mathbb{C}$; that is,

$$\begin{cases} \operatorname{div} |\nabla u|^{p-2} \nabla u = 0 \\ \operatorname{div} |\nabla v|^{p-2} \nabla v = 0 \end{cases}, \quad 1 < p < \infty,\ h = u + \mathrm{i}v$$

such that $h_{|\partial\mathbb{U}} = h_{\circ|\partial\mathbb{U}}$.

The Radó-Kneser-Choquet Theorem ($p = 2$) and the Alessandrini-Sigalotti extension [1] of the Radó-Kneser-Choquet Theorem ($1 < p < \infty$) give a great tool for constructing coordinate-wise p-harmonic homeomorphisms. Namely if, in addition, $h_\circ \colon \partial\mathbb{U} \to \partial\Gamma$ is sense-preserving homeomorphism onto a convex Jordan curve Γ, then h is a $\mathcal{C}^\infty$-diffeomorphism from $\mathbb{U}$ onto the bounded component of $\mathbb{C} \setminus \Gamma$. In particular, $J(z,h) > 0$ in $\mathbb{U}$.

We end this section asking a version of Question 1 for bi-Sobolev mappings. Recall that the inverse of a Sobolev homeomorphism need not be a Sobolev mapping. Even in the one-dimensional case, the homeomorphism $u(x) = x + C(x)$, where C is the usual Cantor function, fails to be absolutely continuous but the inverse of u is a Lipschitz function. Similarly, in higher dimensions, the mapping $g(x_1,\dots,x_2) = (u(x_1), x_2, \dots, x_n)$ provides us with a Lipschitz homeomorphism whose inverse fails to be a Sobolev map. Nevertheless, every homeomorphism and its inverse have bounded variation in the one-dimensional case. The similar duality result holds in the plane, see [24].

Theorem 4.2. *Let $\mathbb{X}, \mathbb{Y}$ be domains in $\mathbb{R}^2$ and suppose that $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a homeomorphism. Then h has bounded variation if and only if the inverse mapping h^{-1} has bounded variation. Moreover, both h and h^{-1} are differentiable almost everywhere.*

The crucial reason why the above mentioned mapping g^{-1} fails to have a Sobolev regularity is that the differential of g^{-1} does not vanish in the zero set of the Jacobian. This means that g^{-1} does not have finite distortion. The following beautiful symmetry result was proved by Hencl and Koskela [23].

Theorem 4.3 ([32])**.** *Let $\mathbb{X}, \mathbb{Y}$ be domains in $\mathbb{R}^2$. Then $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ has finite distortion if the inverse mapping h^{-1} has finite distortion.*

The promised question reads as follows.

Question 2 ([32])**.** A bi-Sobolev homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a mapping of class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \le p < \infty$, whose inverse $h^{-1}\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ belongs to a Sobolev class $\mathscr{W}^{1,q}(\mathbb{Y}, \mathbb{X})$, $1 \le q < \infty$. Can h be approximated by bi-Sobolev diffeomorphisms $\{h_\ell\}$ so that $h_\ell \to h$ in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ and $h_\ell^{-1} \to h^{-1}$ in $\mathscr{W}^{1,q}(\mathbb{Y}, \mathbb{X})$?

5. Two proofs of Schottky's theorem

Here we give two different proofs of Theorem 2.1. The first one uses free-Lagrangians together with sharp estimates. These techniques were developed to study minimization problems between circular annuli §2.1 and §2.3.

5.1. Free Lagrangians. Let $\mathbb{A} = A(r, R)$ and $\mathbb{A}^* = A(r_*, R_*)$ be two circular annuli in $\mathbb{C}$. We shall work with one particular homotopy class $\mathcal{F}(\mathbb{A}, \mathbb{A}^*)$ of $\mathscr{W}^{1,2}$-homeomorphisms $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$. Let $\mathcal{F}_+(\mathbb{A}, \mathbb{A}^*)$ be the class of orientation preserving homeomorphisms $h\colon \mathbb{A} \to \mathbb{A}^*$ in the Sobolev class $\mathscr{W}^{1,2}_{\text{loc}}(\mathbb{A}, \mathbb{A}^*)$ which also preserve the order of the boundary components; that is, $|h(z)| = r_*$ for $|z| = r$ and $|h(z)| = R_*$ for $|z| = R$. Similarly, $\mathcal{F}_-(\mathbb{A}, \mathbb{A}^*)$ is the class of orientation preserving homeomorphisms $h\colon \mathbb{A} \to \mathbb{A}^*$ in $\mathscr{W}^{1,2}_{\text{loc}}(\mathbb{A}, \mathbb{A}^*)$ which reverse the order of the boundary components; that is, $|h(z)| = R_*$ for $|z| = r$ and $|h(z)| = r_*$ for $|z| = R$.

In this context a free Lagrangian refers to a differential 2-form $L(x,h,Dh)\,\mathrm{d}x$, formulated for $h \in \mathcal{F}_\pm(\mathbb{A},\mathbb{A}^*)$, whose integral over $\mathbb{A}$ does not depend on a particular choice of the mapping $h \in \mathcal{F}_\pm(\mathbb{A},\mathbb{A}^*)$. Naturally, polar coordinates

$$x = \rho\,\mathrm{e}^{\mathrm{i}\theta}, \quad r < \rho < R \ \text{ and } \ 0 \le \theta < 2\pi \tag{5.1}$$

are best suited for dealing with mappings of planar annuli. The radial and tangential derivatives of $h\colon \mathbb{A} \to \mathbb{A}^*$ are defined by

$$h_N(x) = \frac{\partial h(\rho\,\mathrm{e}^{\mathrm{i}\theta})}{\partial \rho}, \quad \rho = |x| \tag{5.2}$$

and

$$h_T(x) = \frac{1}{\rho}\frac{\partial h(\rho\,\mathrm{e}^{\mathrm{i}\theta})}{\partial \theta}, \quad \rho = |x|. \tag{5.3}$$

For a general Sobolev mapping we have the formula

$$J(x,h) = \operatorname{Im}(h_T \overline{h_N}) \le |h_T|\,|h_N|. \tag{5.4}$$

We shall make use of three free Lagrangians.

(i) Pullback of a form in $\mathbb{A}^*$ via a given mapping $h \in \mathcal{F}(\mathbb{A},\mathbb{A}^*)$;

$$L(x,h,Dh)\,\mathrm{d}x = N(|h|)J(x,h)\,\mathrm{d}x, \quad \text{where } N \in \mathcal{L}^1(r_*,R_*).$$

Thus, for all $h \in \mathcal{F}(\mathbb{A},\mathbb{A}^*)$ we have

$$\int_{\mathbb{A}} L(x,h,Dh)\,\mathrm{d}x = \int_{\mathbb{A}^*} N(|y|)\,\mathrm{d}y = 2\pi\int_{r_*}^{R_*} N(\tau)\tau\,\mathrm{d}\tau.$$

(ii) A radial free Lagrangian

$$L(x,h,Dh)\,\mathrm{d}x = A(|h|)\frac{|h|_N}{|x|}\,\mathrm{d}x, \quad \text{where } A \in \mathcal{L}^1(r_*,R_*).$$

Thus, for all $h \in \mathcal{F}_\pm(\mathbb{A},\mathbb{A}^*)$ we have

$$\int_{\mathbb{A}} L(x,h,Dh)\,\mathrm{d}x = 2\pi\int_r^R A(|h|)\frac{\partial |h|}{\partial \rho}\,\mathrm{d}\rho = \pm 2\pi\int_{r_*}^{R_*} A(\tau)\,\mathrm{d}\tau.$$

(iii) A tangential free Lagrangian

$$L(x,h,Dh) = B(|x|)\operatorname{Im}\frac{h_T}{h}, \quad \text{where } B \in \mathcal{L}^1(r,R).$$

Thus, for all $h \in \mathcal{F}_\pm(\mathbb{A},\mathbb{A}^*)$ we have

$$\int_{\mathbb{A}} L(x,h,Dh)\,\mathrm{d}x = \int_r^R B(t)\left(\int_{|x|=t}\frac{\partial \operatorname{Arg} h}{\partial \theta}\,\mathrm{d}\theta\right)\mathrm{d}t = \pm 2\pi\int_r^R B(t)\,\mathrm{d}t.$$

5.2. Proof of the theorem. This proof is from [5]. The analytic description of conformality goes via the Cauchy-Riemann equations which we may state in polar coordinates

$$\frac{1}{\rho}\frac{\partial h}{\partial \theta} = \mathrm{i}\frac{\partial h}{\partial \rho}, \text{ equivalently } h_T(z) = \mathrm{i}h_N(z) \text{ for a.e. } z = \rho\,\mathrm{e}^{\mathrm{i}\theta}. \tag{5.5}$$

Therefore, via (5.4), we have

$$J(z,h) = |h_N|^2 = |h_T|^2. \tag{5.6}$$

Suppose that h belongs to $\mathcal{F}_\pm(\mathbb{A},\mathbb{A}^*)$ and satisfies (5.6). Choosing $N(t) = t^{-2}$, $A(t) = t^{-1}$ and $B(t) = 1$ for $t > 0$ in (i)–(iii), we have

$$\begin{aligned} 2\pi\log\frac{R}{r}\cdot 2\pi\log\frac{R_*}{r_*} &= \int_{\mathbb{A}}\frac{\mathrm{d}z}{|z|^2}\cdot\int_{\mathbb{A}}\frac{J(z,h)\,\mathrm{d}z}{|h(z)|^2} \geq \left(\int_{\mathbb{A}}\frac{\sqrt{J(z,h)}\,\mathrm{d}z}{|z|\,|h(z)|}\right)^2 \\ &= \begin{cases} \left(\int_{\mathbb{A}}\left|\frac{h_N}{\rho h}\right|\right)^2 \\ \left(\int_{\mathbb{A}}\left|\frac{h_T}{\rho h}\right|\right)^2 \end{cases} \geq \begin{cases} \left(\int_{\mathbb{A}}\frac{|h|_N}{\rho|h|}\right)^2 \\ \left(\int_{\mathbb{A}}\operatorname{Im}\frac{h_T}{\rho h}\right)^2 \end{cases} = \begin{cases} \left(2\pi\log\frac{R_*}{r_*}\right)^2 \\ \left(2\pi\log\frac{R}{r}\right)^2 \end{cases}. \end{aligned}$$

Hence, a necessary condition for the existence of a conformal map $h\colon \mathbb{A}\to\mathbb{A}^*$ is that $\operatorname{Mod}\mathbb{A} = \operatorname{Mod}\mathbb{A}^*$. Once this condition is satisfied every conformal map $h\colon \mathbb{A}\to\mathbb{A}^*$ must give equality in every step of the above computation. A close inspection of these inequalities reveals that

$$\frac{J(z,h)}{|h(z)|^2} = \frac{m^2}{|z|^2} \quad\text{and}\quad \begin{cases} \left|\frac{h_N}{h}\right| = \pm\frac{h_N}{h} \\ \left|\frac{h_T}{h}\right| = \pm\mathrm{i}\frac{h_T}{h}, \end{cases}$$

where m is a real number. The sign in each equation remains at our choice but must be the same for all points in $\mathbb{A}$. This can easily be summarized in two differential equations

$$\begin{cases} \frac{\partial h}{\partial\rho} = \frac{m}{\rho}h \\ \frac{\partial h}{\partial\theta} = \mathrm{i}mh \end{cases} \quad \text{for some constant } m\in\mathbb{R}. \tag{5.7}$$

Solving these equations poses no difficulty. First the real constant m can be identified from the second equation via the argument principle as follows

$$m = \frac{\partial\operatorname{Arg}h}{\partial\theta} = \pm 1.$$

The plus sign applies when h preserves the order of the boundary components and the minus sign otherwise. Now the general solution takes the form $h(z) = \lambda z^{\pm 1}$, where λ is a complex number whose modulus is uniquely determined by requiring that $|\lambda|r = r_*$ or R_*, respectively. $\square$

5.3. Proof of the theorem and circular means. The second proof gives a flavor of our approach to proving the Nitsche conjecture. The presented proof is from [30]. As in Theorem 3.2 we like to prove a sharp lower estimate for the growth of the circular means of $|h|^2$

$$U(\rho) = \fint_{\mathbb{T}_\rho} |h|^2.$$

Namely, let $\mathcal{A}^*$ be a doubly connected domain whose inner boundary is the unit circle. We do not specify the outer boundary of $\mathcal{A}^*$. Let $\mathfrak{C}(A(1,R),\mathcal{A}^*)$ denote the class of $\mathcal{W}^{1,2}$-homeomorphisms $h\colon A(1,R) \xrightarrow{\text{onto}} \mathcal{A}^*$ satisfying (5.5) which take the unit circle into the unit circle. Our goal is to prove that if $h \in \mathfrak{C}(A(1,R),\mathcal{A}^*)$, then

$$U(\rho) = \fint_{\mathbb{T}_\rho} |h|^2 \geq \rho^2, \quad \dot{U}(\rho) \geq 2\rho, \quad \ddot{U}(\rho) \geq 2. \tag{5.8}$$

Equality occurs, somewhere at $\rho \in (1,R)$, if and only if $h(z) = \lambda z, |\lambda| = 1$.

The classical Schottky theorem follows by imposing the outer boundary condition, $|h(z)| = R_*$ for $|z| = R$, to infer that $R_* \geq R$. This can be reversed via consideration of the inverse conformal mapping, ascertaining Theorem 2.1.

Proof of (5.8)**.** Let us begin with the Laurent expansion

$$h(z) = \sum_{n\in\mathbb{Z}} a_n z^n, \quad 1 < |z| < R. \tag{5.9}$$

The system $\{z^n\}_{n\in\mathbb{Z}}$ is orthogonal on every circle $\mathbb{T}_\rho$, $1<\rho<R$. Thus,

$$U(\rho) = \sum_{n\in\mathbb{Z}} |a_n|^2 \rho^{2n}, \quad 1 < \rho < R.$$

All that matters is to find a certain second order differential operator $\mathcal{L}\colon \mathcal{C}^2(1,R) \to \mathcal{C}(1,R)$, acting on U, that fits into the following scenario:

$$\mathcal{L}[U] \geq 0 \quad \text{with equality if and only if } h(z) = \lambda z,\ |\lambda| = 1. \tag{5.10}$$

A direct computation shows that

$$\mathcal{L}[U] := \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho^3 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{U}{\rho^2}\right)\right] = 4\sum_{n\in\mathbb{Z}} n(n-1)|a_n|^2 \rho^{2n-2} \geq 0. \tag{5.11}$$

Hence,

$$\rho^3 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{U}{\rho^2}\right) \geq \rho^3 \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{U}{\rho^2}\right)\bigg|_{\rho=1} = \dot{U}(1) - 2U(1) = \dot{U}(1) - 2. \tag{5.12}$$

Note that h is $\mathcal{C}^1$-regular up to the inner boundary of $\mathbb{A}$. Even more, since $|h(z)| \equiv 1$ on the unit circle $\mathbb{T}$, it extends as a conformal map slightly inside the unit circle. Being sense preserving homeomorphism, h has the winding number 1 along the unit circle; therefore,

$$1 = \operatorname{Im} \fint_{\mathbb{T}} \frac{h_\theta}{h}.$$

Combining this with (5.5), we have

$$\dot{U}(1) = 2 \operatorname{Re} \fint_{\mathbb{T}} \bar{h}\, h_\rho = 2 \operatorname{Im} \fint_{\mathbb{T}} \bar{h} h_\theta = 2 \operatorname{Im} \fint_{\mathbb{T}} \frac{h_\theta}{h} = 2.$$

By (5.12) we infer that the function $\rho \to \rho^{-2}U(\rho)$ is nondecreasing, and hence $U(\rho) \geq \rho^2$. We actually have slightly stronger inequality $(\rho^{-2}U)' \geq 0$, which yields $\dot{U} \geq 2\rho^{-1}U \geq 2\rho$. Then it follows from (5.11) that $\ddot{U} \geq \rho^{-1}\dot{U} \geq 2$.

Finally, if equality occurs in one of (5.8) for some $1 < \rho < R$, we infer from (5.11) that $a_n = 0$, except for a_0 and a_1, which gives a linear function $h(z) = a_0 + a_1 z$. Since $|h(z)| \equiv 1$ on $\mathbb{T}$, we conclude that $h(z) = \lambda z$ with $|\lambda| = 1$, as desired. □

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