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In: Jiří Rákosník (ed.): Nonlinear Analysis, Function Spaces and Applications, Proceedings of the International School held in Třešť, September 11-17, 2010, Vol. 9. Institute of Mathematics AS CR, Praha, 2011. pp. 217--260.

Persistent URL: http://dml.cz/dmlcz/702642

CALDERÓN-ZYGMUND THEORY WITH NON DOUBLING MEASURES

Xavier Tolsa

Abstract. In this notes we review and prove some results on Calderón-Zygmund theory in $\mathbb{R}^d$ for non doubling measures. To study the behavior of n-dimensional Calderón-Zygmund operators in $\mathbb{R}^d$, instead of the usual doubling condition for a measure μ in $\mathbb{R}^d$, we ask the growth condition $\mu(B(x,r)) \leq c\,r^n$ for all $x \in \mathbb{R}^d$, $r > 0$. Some of the results that we review are the following: the Calderón-Zygmund decomposition, the weak $(1,1)$ boundedness of Calderón-Zygmund operators, Cotlar's inequality, the $T(1)$ theorem, and the definitions of BMO and Hardy spaces. We also describe the relationship between the Cauchy transform and Menger curvature and show its applications to the study of analytic capacity and the so called Painlevé problem.

1. Introduction

In these lecture notes we explain some results on Calderón-Zygmund theory with non doubling measures (also known as *non homogeneous* Calderón-Zygmund theory) in $\mathbb{R}^d$. We also show the application of these results to the so called Painlevé problem.

In recent years it was shown that many results on Calderón-Zygmund theory remain valid if one does not assume that the underlying measure of the space is doubling. Recall that a Borel measure μ on $\mathbb{R}^d$ is said to be doubling if there exists some constant $C > 0$ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \quad \text{for all } x \in \operatorname{supp}(\mu),\ r > 0.$$

2010 *Mathematics Subject Classification.* Primary Please 42B20. Secondary 42B25, 42B30, 42B35, 30C85, 31A15.

Key words and phrases. Non homogeneous Calderón-Zygmund theory; non doubling measures; Calderón-Zygmund decomposition; weak $(1,1)$ boundedness; Cotlar's inequality; $T(1)$ theorem; BMO; Hardy spaces; Cauchy transform; Menger curvature; analytic capacity; Painlevé problem.

One of the main motivations for extending the classical theory to the non doubling context was the solution of several questions related to analytic capacity, like Vitushkin's conjecture [Dd4] or Painlevé's problem [To8]. Other applications of non homogeneous Calderón-Zygmund theory have to do with geometric measure theory and quasiconformal mappings [LSU], [ACTUV].

To study n-dimensional Calderón-Zygmund operators (CZO's) in $\mathbb{R}^d$, with $0 < n \leq d$, we will consider measures μ satisfying the growth condition of degree n

$$\mu(B(x,r)) \leq C_0 r^n \quad \text{for all } x \in \mathbb{R}^d,\ r > 0 \tag{1}$$

(when $n = 1$, we say that μ has linear growth). Let us remark that this is a quite natural condition, because it is necessary for the $L^2(\mu)$ boundedness of any CZO whose kernel $k(x,y)$ satisfies $|k(x,y)| \approx C|x-y|^{-n}$.

One of the main difficulties that arises when one deals with a non doubling measure μ is due to the fact that the non centered maximal Hardy-Littlewood operator

$$M_\mu^{nc} f(x) := \sup\left\{ \frac{1}{\mu(B)} \int_B |f|\, d\mu : B \text{ closed ball, } x \in B \right\}$$

may fail to be of weak type $(1,1)$ (the superindex "nc" stands for non centered). Sometimes the centered version of the operator, that is

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu,$$

is a good substitute of $M_\mu^{nc} f$, because using Besicovitch's covering theorem one can show that M_μ is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$, and in $L^p(\mu)$, for $1 < p \leq \infty$. However, one cannot always use the centered maximal Hardy-Littlewood operator instead of the non centered one. In these cases other arguments are required.

In these lecture notes, we will focus our attention on some basic results of Calderón-Zygmund theory: the weak $(1,1)$ boundedness of CZO's which are bounded in $L^2(\mu)$, using a Calderón-Zygmund type decomposition adapted to the non doubling context; Cotlar's inequality; BMO type spaces; and the $T(1)$ and $T(b)$ theorems. We will give the detailed proofs of the Calderón-Zygmund type decomposition in Section 3, the weak $(1,1)$ boundedness of CZO's in Section 4, and Cotlar's inequality in Section 5. On the other hand, Section 6, which deals with the $T(1)$ theorem, BMO, and H^1, is purely expository. The Cauchy kernel is a very important Calderón-Zygmund kernel, because of its central role in complex analysis. It is also a special kernel due to its relationship with Menger curvature, discovered by MELNIKOV. In Section 7 we explain in detail this relationship, and we give a (new) short prove of the $T(1)$ theorem for the particular case of the Cauchy transform, using a good λ inequality and the connection with Menger curvature. Section 8 consists of a brief introduction to analytic capacity, where we explain the connection between this notion and the weak $(1,1)$ estimates for Cauchy transform. Finally, in Section 9 we show how the results obtained previously are applied to study the Painlevé problem and to prove the semiadditivity of analytic capacity. In particular, we give a detailed proof of the semiadditivity of the capacity γ_+ and its characterization in terms of curvature. However, we do not prove the comparability between the analytic capacity γ and the capacity γ_+. The detailed arguments would lead us too far, out from the scope of these lecture notes.

These notes are not intended to be a survey neither on Calderón-Zygmund theory with non doubling measures nor on the Painlevé problem. We just only describe in some detail, and sometimes prove, some of the results that are important, in our opinion, in connection with Calderón-Zygmund theory with non doubling measures and its applications to the Painlevé problem. Moreover we will concentrate our attention on basic results that can be explained in a rather short minicourse. For this reason, for the sake of simplicity, we will only touch quite superficially some results such as the important $T(b)$ type theorems of NAZAROV, TREIL and VOLBERG in [NTV3], [NTV4] and [NTV5].

Let us remark that some parts of these lecture notes follow quite closely some previous surveys such as [To12] and [To10]. However, the present notes contain more information and details, such as a somewhat new proof of the $T(1)$ theorem for the Cauchy transform in Section 7.

2. PRELIMINARIES

An open ball centered at x with radius r is denoted by $B(x,r)$, and a closed ball by $\bar{B}(x,r)$. By a cube Q we mean a closed cube with sides parallel to the axes. We denote its side length by $\ell(Q)$ and its center by x_Q.

A Radon measure μ on $\mathbb{R}^d$ has growth of degree n (or is of degree n) if there exists some constant C_0 such that $\mu(B(x,r)) \leq C_0 r^n$ for all $x \in \mathbb{R}^d$, $r > 0$. When $n = 1$, we say that μ has linear growth. If there exists some constant C such that

$$C^{-1} r^n \leq \mu(B(x,r)) \leq C r^n \quad \text{for all } x \in \operatorname{supp}(\mu),\ 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)),$$

then we say that μ is n-dimensional AD-regular.

The space of finite complex Radon measures on $\mathbb{R}^d$ is denoted by $M(\mathbb{R}^d)$. This is a Banach space with the norm of the total variation: $\|\mu\| = |\mu|(\mathbb{R}^d)$.

We say that $k(\cdot,\cdot)\colon \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\} \to \mathbb{C}$ is an n-dimensional Calderón-Zygmund kernel if there exist constants $C > 0$ and η, with $0 < \eta \leq 1$, such that the following inequalities hold for all $x, y \in \mathbb{R}^d$, $x \neq y$:

$$
\begin{gathered}
|k(x,y)| \leq \frac{C}{|x-y|^n}, \qquad \text{and} \\
|k(x,y)-k(x',y)| + |k(y,x)-k(y,x')| \leq \frac{C|x-x'|^\eta}{|x-y|^{n+\eta}} \ \text{ if } |x-x'| \leq |x-y|/2.
\end{gathered} \tag{2}
$$

Given a positive or complex Radon measure ν on $\mathbb{R}^d$, we define

$$
T\nu(x) := \int k(x,y)\,d\nu(y), \quad x \in \mathbb{R}^d \setminus \operatorname{supp}(\nu). \tag{3}
$$

We say that T is an n-dimensional Calderón-Zygmund operator (CZO) with kernel $k(\cdot,\cdot)$. The integral in the definition may not be absolutely convergent if $x \in \operatorname{supp}(\mu)$. For this reason, we consider the following ε-truncated operators T_ε, $\varepsilon > 0$:

$$
T_\varepsilon\nu(x) := \int_{|x-y|>\varepsilon} k(x,y)\,d\nu(y), \quad x \in \mathbb{R}^d.
$$

Observe that now the integral on the right-hand side converges absolutely if, for instance, $|\nu|(\mathbb{R}^d) < \infty$.

Given a fixed positive Radon measure μ on $\mathbb{R}^d$ and $f \in L^1_{\text{loc}}(\mu)$, we denote

$$
T_\mu f(x) := T(f\,d\mu)(x), \quad x \in \mathbb{R}^d \setminus \operatorname{supp}(f\,d\mu),
$$

and

$$
T_{\mu,\varepsilon} f(x) := T_\varepsilon(f\,d\mu)(x).
$$

The last definition makes sense for all $x \in \mathbb{R}^d$ if, for example, $f \in L^1(\mu)$. We say that T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$. Analogously, with respect to the boundedness from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. We also say that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$ if there exists some constant C such that for all $\nu \in M(\mathbb{R}^d)$ and all $\lambda > 0$,

$$
\mu\{x \in \mathbb{R}^d : |T_\varepsilon \nu| > \lambda\} \leq \frac{C\|\nu\|}{\lambda}
$$

uniformly on $\varepsilon > 0$.

The Cauchy transform is the CZO on $\mathbb{C}$ originated by the kernel

$$k(x,y) := \frac{1}{y-x}, \quad x, y \in \mathbb{C}.$$

It is denoted by $\mathcal{C}$. That is to say,

$$\mathcal{C}\mu(x) := \int \frac{1}{y-x}\, d\mu(y), \quad x \in \mathbb{C} \setminus \operatorname{supp}(\mu).$$

As usual, in the paper the letter 'C' stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as C_1, retain its value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \le CB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

3. Calderón-Zygmund decomposition

3.1. Doubling cubes. Given $\alpha > 1$ and $\beta > \alpha^n$, we say that a cube Q is (α, β)-doubling if $\mu(\alpha Q) \le \beta\mu(Q)$, where αQ is the cube concentric with Q with side length $\alpha\ell(Q)$. For definiteness, if α and β are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube.

Because μ satisfies the growth condition (1), there are a lot of "big" doubling cubes. To be precise, given any point $x \in \operatorname{supp}(\mu)$ and $c > 0$, there exists some (α, β)-doubling cube Q centered at x with $l(Q) \ge c$. This follows easily from (1) and the fact that $\beta > \alpha^n$. Indeed, if there are no doubling cubes centered at x with $l(Q) \ge c$, then $\mu(\alpha^m Q) > \beta^m \mu(Q)$ for each m, and letting $m \to \infty$ one sees that (1) cannot hold.

Next lemma states that there are a lot of "small" doubling cubes too:

Lemma 1. *Let $\beta > \alpha^d$. If μ is a Radon measure in $\mathbb{R}^d$, then for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence of (α, β)-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$.*

Notice that the statement of the lemma is valid for any Radon measure on $\mathbb{R}^d$. In particular, it is not necessary to assume the growth condition (1).

Proof. Let $Z \subset \mathbb{R}^d$ be the set of points x such that there does not exist a sequence of (α, β)-doubling cubes $\{Q_k\}_{k \ge 0}$ centered at x with side length decreasing to 0; and let $Z_j \subset \mathbb{R}^d$ be the set of points x such that there does not exist any (α, β)-doubling cube Q centered at x with $\ell(Q) \le 2^{-j}$. Clearly, $Z = \bigcup_{j \ge 0} Z_j$. Thus, proving the lemma is equivalent to showing that $\mu(Z_j) = 0$ for every $j \ge 0$.

Let Q_0 be a fixed cube with side length 2^{-j} and let $k \geq 1$ be some integer. For each $z \in Q_0 \cap Z_j$, let Q_z be a cube centered at z with side length $\alpha^{-k}\ell(Q_0)$. Since the cubes $\alpha^h Q_z$ are not (α, β)-doubling for $h = 0, \ldots, k-1$ and $\alpha^k Q_z \subset 2Q_0$, we have

$$\mu(Q_z) \leq \beta^{-1}\mu(\alpha Q_z) \leq \cdots \leq \beta^{-k}\mu(\alpha^k Q_z) \leq \beta^{-k}\mu(2Q_0). \tag{4}$$

By the Besicovitch theorem, there exists a subfamily $\{z_m\} \subset Q_0 \cap Z_j$ such that $Q_0 \cap Z_j \subset \bigcup_m Q_{z_m}$ and moreover $\sum_m \chi_{Q_{z_m}} \leq C_d$. This is a finite family and the number N of points z_m can be easily bounded above as follows: if $\mathcal{L}$ stands for the Lebesgue measure on $\mathbb{R}^d$,

$$N(\alpha^{-k}\ell(Q_0))^d = \sum_{m=1}^{N} \mathcal{L}(Q_{z_m}) \leq C_d \mathcal{L}(2Q_0) = C_d(2\ell(Q_0))^d.$$

Thus,

$$N \leq C_d 2^d \alpha^{kd}.$$

As a consequence, since the family $\{Q_{z_m}\}_{1 \leq m \leq N}$ covers $Q_0 \cap Z_j$, by (4),

$$\mu(Q_0 \cap Z_j) \leq \sum_{m=1}^{N} \mu(Q_z) \leq N\beta^{-k}\mu(2Q_0) \leq C_d 2^d \alpha^{kd}\beta^{-k}\mu(2Q_0).$$

Since $\beta > \alpha^d$, the right-hand side tends to 0 as $k \to \infty$. Therefore $\mu(Q_0 \cap Z_j) = 0$, and since the cube Q_0 is arbitrary, we are done. □

Remark 2. Given $f \in L^1_{loc}(\mu)$, by the Lebesgue differentiation theorem, for μ-almost all $x \in \mathbb{R}^d$, every sequence of $(2, 2^{d+1})$-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ satisfies

$$\lim_{k\to\infty} \frac{1}{\mu(Q_k)} \int_{Q_k} f \, d\mu = f(x). \tag{5}$$

By the preceding lemma, for μ-a.e. $x \in \mathbb{R}^d$, there exists a sequence of $(2, 2^{d+1})$-doubling cubes $\{Q_k\}_k$ with $\ell(Q_k) \to 0$ satisfying (5). In particular, for any fixed $\lambda > 0$, for μ-almost all $x \in \mathbb{R}^d$ such that $|f(x)| > \lambda$, there exists a sequence of cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ such that

$$\limsup_{k\to\infty} \frac{1}{\mu(2Q_k)} \int_{Q_k} |f| \, d\mu > \frac{\lambda}{2^{d+1}}.$$

3.2. Calderón-Zygmund decomposition.

Lemma 3 (Calderón-Zygmund decomposition, [To3]). *Assume that μ satisfies* (1). *For any $f \in L^1(\mu)$ and any $\lambda > 0$ (with $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$ if $\|\mu\| < \infty$) we have:*

(a) *There exists a family of almost disjoint cubes $\{Q_i\}_i$ (i.e., $\sum_i \chi_{Q_i} \leq C$) such that*

$$\frac{1}{\mu(2Q_i)} \int_{Q_i} |f|\, d\mu > \frac{\lambda}{2^{d+1}}, \tag{6}$$

$$\frac{1}{\mu(2\eta Q_i)} \int_{\eta Q_i} |f|\, d\mu \leq \frac{\lambda}{2^{d+1}} \quad \text{for } \eta > 2, \tag{7}$$

$$|f| \leq \lambda \quad \text{a.e. } \mu \text{ on } \mathbb{R}^d \setminus \bigcup_i Q_i. \tag{8}$$

(b) *For each i, let R_i be a $(6, 6^{n+1})$-doubling cube concentric with Q_i, with $l(R_i) > 4l(Q_i)$ and denote $w_i = \frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}$. Then, there exists a family of functions φ_i with $\operatorname{supp}(\varphi_i) \subset R_i$ and with constant sign satisfying*

$$\int \varphi_i\, d\mu = \int_{Q_i} f w_i\, d\mu, \tag{9}$$

$$\sum_i |\varphi_i| \leq B\lambda \tag{10}$$

(where B is some constant), and

$$\|\varphi_i\|_{L^\infty(\mu)} \mu(R_i) \leq C \int_{Q_i} |f|\, d\mu. \tag{11}$$

Proof. (a) Recall that, by Remark 2, for μ-almost all $x \in \mathbb{R}^d$ such that $|f(x)| > \lambda$, there exists some cube Q_x satisfying

$$\frac{1}{\mu(2Q_x)} \int_{Q_x} |f|\, d\mu > \frac{\lambda}{2^{d+1}} \tag{12}$$

and such that if Q'_x is centered at x with $l(Q'_x) > 2l(Q_x)$, then

$$\frac{1}{\mu(2Q'_x)} \int_{Q'_x} |f|\, d\mu \leq \frac{\lambda}{2^{d+1}}.$$

Now we can apply Besicovitch's covering theorem (see Remark 4 below) to get an almost disjoint subfamily of cubes $\{Q_i\}_i \subset \{Q_x\}_x$ satisfying (6), (7) and (8).

(b) Assume first that the family of cubes $\{Q_i\}_i$ is finite. Then we may suppose that this family of cubes is ordered in such a way that the sizes of the cubes R_i are non decreasing (i.e. $l(R_{i+1}) \geq l(R_i)$). The functions φ_i that we will construct will be of the form $\varphi_i = \alpha_i \chi_{A_i}$, with $\alpha_i \in \mathbb{R}$ and $A_i \subset R_i$. We set $A_1 = R_1$ and $\varphi_1 = \alpha_1 \chi_{R_1}$, where the constant α_1 is chosen so that $\int_{Q_1} f w_1 \, d\mu = \int \varphi_1 \, d\mu$.

Suppose that $\varphi_1, \ldots, \varphi_{k-1}$ have been constructed, satisfy (9) and

$$\sum_{i=1}^{k-1} |\varphi_i| \leq B\lambda,$$

where B is some constant which will be fixed below.

Let $R_{s_1}, \ldots, R_{s_m}$ be the subfamily of $R_1, \ldots, R_{k-1}$ such that $R_{s_j} \cap R_k \neq \emptyset$. As $l(R_{s_j}) \leq l(R_k)$ (because of the non decreasing sizes of R_i), we have $R_{s_j} \subset 3R_k$. Taking into account that for $i = 1, \ldots, k-1$,

$$\int |\varphi_i| \, d\mu \leq \int_{Q_i} |f| \, d\mu$$

by (9), and using that R_k is $(6, 6^{n+1})$-doubling and (7), we get

$$\begin{aligned} \sum_j \int |\varphi_{s_j}| \, d\mu &\leq \sum_j \int_{Q_{s_j}} |f| \, d\mu \\ &\leq C \int_{3R_k} |f| \, d\mu \leq C\lambda\mu(6R_k) \leq C_2 \lambda \mu(R_k). \end{aligned}$$

Therefore,

$$\mu\bigl\{\textstyle\sum_j |\varphi_{s_j}| > 2C_2\lambda\bigr\} \leq \frac{\mu(R_k)}{2}.$$

So we set

$$A_k = R_k \cap \bigl\{\textstyle\sum_j |\varphi_{s_j}| \leq 2C_2\lambda\bigr\},$$

and then $\mu(A_k) \geq \mu(R_k)/2$.

The constant α_k is chosen so that for $\varphi_k = \alpha_k \chi_{A_k}$ we have $\int \varphi_k \, d\mu = \int_{Q_k} f w_k \, d\mu$. Then we obtain

$$|\alpha_k| \leq \frac{1}{\mu(A_k)} \int_{Q_k} |f| \, d\mu \leq \frac{2}{\mu(R_k)} \int_{\frac{1}{2}R_k} |f| \, d\mu \leq C_3 \lambda$$

(this calculation also applies to $k = 1$). Thus,

$$|\varphi_k| + \sum_j |\varphi_{s_j}| \le (2C_2 + C_3)\lambda.$$

If we choose $B = 2C_2 + C_3$, (10) follows.

Now it is easy to check that (11) also holds. Indeed we have

$$\|\varphi_i\|_{L^\infty(\mu)}\mu(R_i) \le C|\alpha_i|\mu(A_i) = C\Big|\int_{Q_i} f w_i \, d\mu\Big| \le C\int_{Q_i} |f| \, d\mu.$$

Suppose now that the collection of cubes $\{Q_i\}_i$ is not finite. For each fixed N we consider the family of cubes $\{Q_i\}_{1\le i\le N}$. Then, as above, we construct functions $\varphi_1^N, \dots, \varphi_N^N$ with $\operatorname{supp}(\varphi_i^N) \subset R_i$ satisfying

$$\int \varphi_i^N \, d\mu = \int_{Q_i} f w_i \, d\mu,$$

$$\sum_{i=1}^N |\varphi_i^N| \le B\lambda$$

and

$$\|\varphi_i^N\|_{L^\infty(\mu)}\mu(R_i) \le C\int_{Q_i} |f| \, d\mu.$$

Notice that the sign of φ_i^N equals the sign of $\int f w_i \, d\mu$ and so it does not depend on N.

Then there is a subsequence $\{\varphi_1^k\}_{k\in I_1}$ which is convergent in the weak $*$ topology of $L^\infty(\mu)$ to some function $\varphi_1 \in L^\infty(\mu)$. Now we can consider a subsequence $\{\varphi_2^k\}_{k\in I_2}$ with $I_2 \subset I_1$ which is also convergent in the weak $*$ topology of $L^\infty(\mu)$ to some function $\varphi_2 \in L^\infty(\mu)$. In general, for each j we consider a subsequence $\{\varphi_j^k\}_{k\in I_j}$ with $I_j \subset I_{j-1}$ that converges in the weak $*$ topology of $L^\infty(\mu)$ to some function $\varphi_j \in L^\infty(\mu)$. It is easily checked that the functions φ_j satisfy the required properties. □

Remark 4. Recall that Besicovitch's covering theorem asserts that if $\Omega \subset \mathbb{R}^d$ is a *bounded* set and for each $x \in \Omega$ there is a cube Q_x centered at x, then there exists a family of cubes $\{Q_{x_i}\}_i$ with finite overlap, that is $\sum_i \chi_{Q_i} \le C$, which covers Ω.

In (a) of the preceeding proof we have applied Besicovitch's covering theorem to $\Omega = \{x : |f(x)| > \lambda\}$. However this set may be unbounded, and the boundedness property is a necessary assumption in Besicovitch's theorem (example: take $\Omega = [0, +\infty) \subset \mathbb{R}$ and consider $Q_x = [0, 2x]$ for all $x \in \Omega$).

We can solve this problem using different arguments. Let us sketch a possible solution to this little trouble. Consider a cube Q_0 centered at 0 big enough so that $2^{d+1}\|f\|_{L^1(\mu)}/\mu(Q_0) < \lambda$. So for any cube Q containing Q_0 we will have

$$2^{d+1}\|f\|_{L^1(\mu)}/\mu(Q) < \lambda. \tag{13}$$

For $m \geq 0$ we set $Q_m := \left(\frac{5}{4}\right)^m Q_0$. For each m we can apply Besicovitch's covering theorem to the annulus $Q_m \setminus Q_{m-1}$ (we take $Q_{-1} := \emptyset$), with cubes Q_x centered at $x \in \operatorname{supp}(\mu) \cap (Q_m \setminus Q_{m-1})$ as in (a) of the proof above, satisfying (12).

In this argument we have to be careful with the overlapping among the cubes belonging to coverings of different annuli. Indeed, there exist some fixed constants N and N' such that if $m \geq N'$, for $x \in \operatorname{supp}(\mu) \cap (Q_m \setminus Q_{m-1})$ we have

$$Q_x \subset Q_{m+N} \setminus Q_{m-N}. \tag{14}$$

Otherwise, it is easily seen that $\ell(Q_x) > \frac{3}{4}\ell(Q_m)$, choosing N big enough. It follows that $Q_0 \subset 2Q_x$ since $\ell(Q_0) \ll \ell(Q_m)$ for N' big enough too. This cannot happen because then $2Q_x$ satisfies (13), which contradicts (12).

Because of (14), the covering made up of squares belonging to the Besicovitch coverings of different annuli $Q_m \setminus Q_{m-1}$, $m \geq 0$, will have finite overlap.

Notice that in this argument, it is essential the fact that in (12) we are not dividing by $\mu(Q_x)$, but by $\mu(2Q_x)$.

In the next lemma we prove a very useful estimate involving non doubling squares which relies on the idea that the mass μ which lives on non doubling squares must be small.

Lemma 5. *If $Q \subset R$ are concentric cubes such that there are no (α, β)-doubling cubes (with $\beta > \alpha^n$) of the form $\alpha^k Q$, $k \geq 0$, with $Q \subset \alpha^k Q \subset R$, and x_Q stands for the center of Q, then*

$$\int_{R \setminus Q} \frac{1}{|x - x_Q|^n} \, d\mu(x) \leq C_1,$$

where C_1 depends only on α, β, n, d and C_0.

Proof. Let N be the least integer such that $R \subset \alpha^N Q$. For $0 \leq k \leq N$ we have $\mu(\alpha^k Q) \leq \mu(\alpha^N Q)/\beta^{N-k}$. Then,

$$\begin{aligned}\int_{R\setminus Q} \frac{1}{|x-x_Q|^n}\, d\mu(x) &\leq \sum_{k=1}^{N} \int_{\alpha^k Q\setminus \alpha^{k-1}Q} \frac{1}{|x-x_Q|^n}\, d\mu(x)\\ &\leq C\sum_{k=1}^{N} \frac{\mu(\alpha^k Q)}{\ell(\alpha^k Q)^n}\\ &\leq C\sum_{k=1}^{N} \frac{\beta^{k-N}\mu(\alpha^N Q)}{\alpha^{(k-N)n}\ell(\alpha^N Q)^n}\\ &\leq C\frac{\mu(\alpha^N Q)}{\ell(\alpha^N Q)^n}\sum_{j=0}^{\infty}\Big(\frac{\alpha^n}{\beta}\Big)^j\\ &\leq C.\end{aligned}$$

□

4. Weak (1,1) boundedness of Calderón-Zygmund operators

Theorem 6. *Let μ be a Radon measure on $\mathbb{R}^d$ satisfying the growth condition* (1). *If T is an n-dimensional Calderón-Zygmund operator which is bounded in $L^2(\mu)$, then it is also bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. In particular, T_μ is of weak type* $(1,1)$.

The preceding result was first obtained in [NTV2], although a previous proof valid only for the Cauchy transform appeared in [To1]. The proof below is from [To4].

Proof. We will show that T_μ is of weak type $(1,1)$. By similar arguments, one gets that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. In this case, one has to use a version of the Calderón-Zygmund decomposition in the lemma above suitable for complex measures (see the end of the proof for more details).

Let $f \in L^1(\mu)$ and $\lambda > 0$. It is straightforward to check that we may assume $\lambda > 2^{d+1}\|f\|_{L^1(\mu)}/\|\mu\|$. Let $\{Q_i\}_i$ be the almost disjoint family of cubes of Lemma 3. Let R_i be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_i$, $k \geq 1$. Then we can write $f = g + b$, with

$$g = f\chi_{\mathbb{R}^d\setminus \bigcup_i Q_i} + \sum_i \varphi_i$$

and

$$b = \sum_i b_i := \sum_i (w_i f - \varphi_i),$$

where the functions φ_i satisfy (9), (10) (11) and $w_i = \frac{\chi_{Q_i}}{\sum_k \chi_{Q_k}}$.

By (6) we have

$$\mu\Big(\bigcup_i 2Q_i\Big) \leq \frac{C}{\lambda}\sum_i \int_{Q_i} |f|\,d\mu \leq \frac{C}{\lambda}\int |f|\,d\mu.$$

So we have to show that

$$\mu\Big\{x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i : |T_{\mu,\varepsilon}f(x)| > \lambda\Big\} \leq \frac{C}{\lambda}\int |f|\,d\mu. \tag{15}$$

Since $\int b_i\,d\mu = 0$, $\operatorname{supp}(b_i) \subset R_i$ and $\|b_i\|_{L^1(\mu)} \leq C\int_{Q_i}|f|\,d\mu$, using condition 2 in the definition of a Calderón-Zygmund kernel (which implies Hörmander's condition), we get

$$\int_{\mathbb{R}^d\setminus 2R_i} |T_{\mu,\varepsilon}b_i|\,d\mu \leq C\int |b_i|\,d\mu \leq C\int_{Q_i}|f|\,d\mu.$$

Let us see that

$$\int_{2R_i\setminus 2Q_i} |T_{\mu,\varepsilon}b_i|\,d\mu \leq C\int_{Q_i}|f|\,d\mu, \tag{16}$$

too. On the one hand, by (11) and using the $L^2(\mu)$ boundedness of T and that R_i is $(6, 6^{n+1})$-doubling we get

$$\begin{aligned}\int_{2R_i} |T_{\mu,\varepsilon}\varphi_i|\,d\mu &\leq \left(\int_{2R_i}|T_{\mu,\varepsilon}\varphi_i|^2\,d\mu\right)^{1/2}\mu(2R_i)^{1/2}\\ &\leq C\left(\int|\varphi_i|^2\,d\mu\right)^{1/2}\mu(R_i)^{1/2}\\ &\leq C\int_{Q_i}|f|\,d\mu.\end{aligned}$$

On the other hand, since $\operatorname{supp}(w_i f) \subset Q_i$, if $x \in 2R_i \setminus 2Q_i$, then

$$|T_{\mu,\varepsilon}(w_i f)(x)| \leq C\int_{Q_i}\frac{|f|}{|x - x_{Q_i}|^n}\,d\mu,$$

and so

$$\int_{2R_i\setminus 2Q_i}|T_{\mu,\varepsilon}(w_i f)|\,d\mu \leq C\int_{2R_i\setminus 2Q_i}\frac{1}{|x - x_{Q_i}|^n}\,d\mu(x)\int_{Q_i}|f|\,d\mu.$$

By Lemma 5, the first integral on the right-hand side is bounded by some constant independent of Q_i and R_i, since there are no $(6, 6^{n+1})$-doubling cubes of the form $6^k Q_i$ between $6Q_i$ and R_i. Therefore, (16) holds.

Then we have

$$\begin{aligned}\int_{\mathbb{R}^d \setminus \bigcup_k 2Q_k} |T_{\mu,\varepsilon} b| \, d\mu &\leq \sum_i \int_{\mathbb{R}^d \setminus \bigcup_k 2Q_k} |T_{\mu,\varepsilon} b_i| \, d\mu \\ &\leq C \sum_i \int_{Q_i} |f| \, d\mu \leq C \int |f| \, d\mu.\end{aligned}$$

Therefore,

$$\mu\Big\{x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i : |T_{\mu,\varepsilon} b(x)| > \lambda\Big\} \leq \frac{C}{\lambda} \int |f| \, d\mu. \tag{17}$$

The corresponding integral for the function g is easier to estimate. Taking into account that $|g| \leq C\lambda$, we get

$$\mu\Big\{x \in \mathbb{R}^d \setminus \bigcup_i 2Q_i : |T_{\mu,\varepsilon} g(x)| > \lambda\Big\} \leq \frac{C}{\lambda^2} \int |g|^2 \, d\mu \leq \frac{C}{\lambda} \int |g| \, d\mu. \tag{18}$$

Also, we have

$$\begin{aligned}\int |g| \, d\mu &\leq \int_{\mathbb{R}^d \setminus \bigcup_i Q_i} |f| \, d\mu + \sum_i \int |\varphi_i| \, d\mu \\ &\leq \int |f| \, d\mu + \sum_i \int_{Q_i} |f| \, d\mu \\ &\leq C \int |f| \, d\mu.\end{aligned}$$

Now, by (17) and (18) we get (15).

If we want to show that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, then in Lemma 3 and in the arguments above $f \, d\mu$ must be substituted by $d\nu$, with $\nu \in M(\mathbb{R}^d)$, and $|f| \, d\mu$ by $d|\nu|$. Also, condition (8) of Lemma 3 should be stated as "on $\mathbb{R}^d \setminus \bigcup_i Q_i$, ν is absolutely continuous with respect to μ, that is $\nu = f \, d\nu$, and moreover $|f(x)| \leq \lambda$ a.e. (μ) $x \in \mathbb{R}^d \setminus \bigcup_i Q_i$". With other minor changes, the arguments and estimates above work in this situation too. $\square$

5. Cotlar's inequality

This inequality involves some maximal operators which we proceed to define. The *centered maximal Hardy-Littlewood operator* applied to $\nu \in M(\mathbb{R}^d)$ is, as usual,

$$M_\mu \nu(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} d|\nu|.$$

A useful variant of this operator is the following:

$$\begin{aligned}&\widetilde{M}_\mu \nu(x)\\ &\quad = \sup\Big\{ \frac{1}{\mu(B(x,r))} \int_{B(x,r)} d|\nu| : r > 0,\ \mu(B(x,5r)) \leq 5^{d+1} \mu(B(x,r)) \Big\}.\end{aligned}$$

The non centered version of $\widetilde{M}_\mu$ is

$$N_\mu \nu(x) = \sup\Big\{ \frac{1}{\mu(B)} \int_B d|\nu| : B \text{ closed ball},\ x \in B,\ \mu(5B) \leq 5^{d+1}\mu(B) \Big\}.$$

For $f \in L^1_{\text{loc}}(\mu)$ we set $M_\mu f \equiv M_\mu(f d\mu)$, $\widetilde{M}_\mu f \equiv \widetilde{M}_\mu(f d\mu)$, and $N_\mu f \equiv N_\mu(f d\mu)$. The operators M_μ and $\widetilde{M}_\mu$ are bounded in $L^p(\mu)$, and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. This fact can be proved using Besicovitch's covering theorem for M_μ and $\widetilde{M}_\mu$, and Vitali's covering theorem with balls $B(x,5r)$ in the case of N_μ.

If T is a CZO, the *maximal operator* T_* is

$$T_* \nu(x) = \sup_{\varepsilon>0} |T_\varepsilon \nu(x)| \quad \text{for } \nu \in M(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

and the *δ-truncated maximal operator* $T_{*,\delta}$ is

$$T_{*,\delta} \nu(x) = \sup_{\varepsilon>\delta} |T_\varepsilon \nu(x)| \quad \text{for } \nu \in M(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

We also set $T_{\mu,*} f \equiv T_*(f\, d\mu)$ and $T_{\mu,*,\delta} f \equiv T_{*,\delta}(f\, d\mu)$ for $f \in L^1_{\text{loc}}(\mu)$.

Theorem 7 (Cotlar's inequality)**.** *Let μ be a positive Radon measure on $\mathbb{R}^d$ with growth of degree n. If the T is an n-dimensional CZO bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, then for $0 < s \leq 1$ we have*

$$T_{*,\delta} \nu(x) \leq C_s \Big(\Big(\widetilde{M}_\mu (|T_\delta \nu|^s)(x) \Big)^{1/s} + M_\mu \nu(x) \Big) \textit{ for } \nu \in M(\mathbb{R}^d),\ x \in \mathbb{R}^d, \quad (19)$$

where C_s depends only on the constant C_0 in (1), s, n, d, and the norm of the T_δ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$.

If we assume that that there exists $T\nu$ defined in a reasonable sense (for instance, as a principal value, or as a some kind of weak limit), then we get the more classical version of Cotlar's inequality

$$T_*\nu(x) \leq C_s\Big(\big(\widetilde{M}_\mu(|T\nu|^s)(x)\big)^{1/s} + M_\mu\nu(x)\Big).$$

Cotlar's inequality with non doubling measures is due to Nazarov, Treil and Volberg [NTV2], although not in the form stated above, which is from [To2].

To prove Theorem 7 we will need some lemmas. The first one is Kolmogorov's inequality whose proof can be found in [Ma2, p. 299], for example.

Lemma 8. *Let μ be a positive Radon measure on $\mathbb{R}^d$ and $f\colon \mathbb{R}^d \to \mathbb{C}$ a Borel function in $L^{1,\infty}(\mu)$. Then for $0 < s < 1$ and for any μ-measurable set $A \subset \mathbb{R}^d$ with $\mu(A) < \infty$,*

$$\left(\frac{1}{\mu(A)}\int_A |f|^s d\mu\right)^{1/s} \leq (1-s)^{-1/s}\frac{\|f\|_{L^{1,\infty}(\mu)}}{\mu(A)}.$$

Also, we will need the following result (notice the resemblances with Lemma 5).

Lemma 9. *Let $0 < r < R$, with $R = 5^N r$, and take $\beta > 5^n$. If*

$$\beta\mu(B(x,5^k r)) \leq \mu(B(x,5^{k+1}r)) \quad \text{for } k = 1,\dots,N-1,$$

then we have

$$|T_R\nu(x) - T_r\nu(x)| \leq \frac{\mu(B(x,R))}{R^n}M_\mu\nu(x),$$

for each $\nu \in M(\mathbb{R}^d)$.

Proof. We set $B_k = B(x,5^k r)$. Then,

$$\begin{aligned}
|T_R\nu(x) - T_r\nu(x)| &= \Big|\int_{r<|y-x|\leq 5^N r} k(x,y)\,d\nu(y)\Big| \\
&\lesssim \sum_{k=1}^N \int_{5^{k-1}r<|y-x|\leq 5^k r} \frac{1}{|y-x|^n}\,d|\nu|(y) \qquad (20)\\
&\lesssim \sum_{k=1}^N \frac{|\nu|(B_k)}{(5^k r)^n} \lesssim \sum_{k=1}^N \frac{\mu(B_k)}{(5^k r)^n}M_\mu\nu(x).
\end{aligned}$$

We have

$$\mu(B_1) \leq \frac{1}{\beta}\mu(B_2) \leq \cdots \leq \frac{1}{\beta^{N-1}}\mu(B_N).$$

Thus

$$\frac{\mu(B_1)}{(5r)^n} \leq \frac{5^n}{\beta}\frac{\mu(B_2)}{(5^2r)^n} \leq \cdots \leq \Big(\frac{5^n}{\beta}\Big)^{N-1}\frac{\mu(B_N)}{(5^Nr)^n}.$$

Since $\frac{5^n}{\beta} < 1$, we get

$$\sum_{k=1}^N \frac{\mu(B_k)}{(5^kr)^n} \leq C\frac{\mu(B_N)}{(5^Nr)^n},$$

and the lemma follows from (20). □

Combining Lemma 9 with the usual arguments we are going to prove Cotlar's inequality (19).

Proof of Theorem 7. Let $\varepsilon > \delta$ and $x \in \mathbb{R}^d$. Since μ has growth of degree n, there exists some $k \geq 1$ such that

$$\mu(B(x, 5^m\varepsilon)) \leq 5^{d+1}\mu(B(x, 5^{m-1}\varepsilon)) \tag{21}$$

(see Subsection 3.1). We assume that m is the least integer ≥ 1 such that (21) holds. Set $\varepsilon' = 5^m\varepsilon$. By Lemma 9,

$$|T_\varepsilon\nu(x) - T_{\varepsilon'/5}\nu(x)| \leq CM_\mu\nu(x).$$

Also, it is straightforward to check that $|T_{\varepsilon'/5}\nu(x) - T_{\varepsilon'}\nu(x)| \leq CM_\mu\nu(x)$. Therefore,

$$|T_\varepsilon\nu(x) - T_{\varepsilon'}\nu(x)| \leq CM_\mu\nu(x).$$

Thus it only remains to show that

$$|T_{\varepsilon'}\nu(x)| \leq C_s\Big(\big(\widetilde{M}_\mu(|T_\delta\nu|^s)(x)\big)^{1/s} + M_\mu\nu(x)\Big). \tag{22}$$

Since

$$\mu(B(x, \varepsilon')) \leq 5^{d+1}\mu(B(x, \varepsilon'/5)), \tag{23}$$

we can apply the usual argument, as in [Ma2, pp. 299–300], to prove (22). We set

$$d\nu_1 = \chi_{B(x,\varepsilon')}\,d\nu, \quad d\nu_2 = d\nu - d\nu_1.$$

For $y \in B(x,\varepsilon'/5)$, since $\varepsilon' > 5\delta$ we have $T_{\varepsilon'}\nu_2(x) = T_\delta\nu_2(x) = T\nu_2(x)$ and $T_\delta\nu_2(y) = T\nu_2(y)$. Using (1) it is easy to check that $|T_\delta\nu_2(y) - T_\delta\nu_2(x)| \leq CM_\mu\nu(x)$. Therefore,

$$\begin{aligned} |T_{\varepsilon'}\nu(x)| = |T_\delta\nu_2(x)| &\leq |T_\delta\nu_2(y)| + C_2M_\mu\nu(x) \\ &\leq |T_\delta\nu_1(y)| + |T_\delta\nu(y)| + C_2M_\mu\nu(x). \end{aligned} \tag{24}$$

Assume first $s = 1$. If $T_{\varepsilon'}\nu(x) \neq 0$, let $0 < \lambda < |T_{\varepsilon'}\nu(x)|$. For $y \in B(x,\varepsilon'/5)$, by (24) either $C_2M_\mu\nu(x) > \lambda/3$ or $|T_\delta\nu(y)| > \lambda/3$ or $|T_\delta\nu_1(y)| > \lambda/3$. Therefore, either

$$\lambda < 3C_2M_\mu\nu(x),$$

or

$$\begin{aligned} B(x,\varepsilon'/5) = \{&y \in B(x,\varepsilon'/5) : |T_\delta\nu(y)| > \lambda/3\} \\ &\cup \{y \in B(x,\varepsilon'/5) : |T_\delta\nu_1(y)| > \lambda/3\}. \end{aligned}$$

We have

$$\begin{aligned} \mu\{y \in B(x,\varepsilon'/5) : |T_\delta\nu(y)| > \lambda/3\} &\leq \frac{3}{\lambda}\int_{B(x,\varepsilon'/5)} |T_\delta\nu|\,d\mu \\ &\leq \frac{3}{\lambda}\mu(B(x,\varepsilon'/5))\widetilde{M}_\mu(T_\delta\nu)(x), \end{aligned}$$

and by the boundedness of T_δ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$ and (23),

$$\begin{aligned} \mu\{y \in B(x,\varepsilon'/5) : |T_\delta\nu_1(y)| > \lambda/3\} &\lesssim \frac{\|\nu_1\|}{\lambda} \\ &= \frac{|\nu|(B(x,\varepsilon'))}{\lambda} \\ &\lesssim \frac{\mu(B(x,\varepsilon'/5))}{\lambda}M_\mu\nu(x). \end{aligned}$$

In any case we obtain $\lambda < 3\widetilde{M}_\mu(T_\delta\nu)(x) + CM_\mu\nu(x)$. Since this holds for $0 < \lambda < |T_{\varepsilon'}\nu(x)|$, (22) follows when $s = 1$.

Assume now $0 < s < 1$. From (24) we get

$$|T_{\varepsilon'}\nu(x)|^s \leq |T_\delta\nu_1(y)|^s + |T_\delta\nu(y)|^s + CM_\mu\nu(x)^s.$$

Integrating with respect to μ and $y \in B(x,\varepsilon'/5)$, dividing by $\mu(B(x,\varepsilon'/5))$ and raising to the power $1/s$ we obtain

$$\begin{aligned} |T_{\varepsilon'}\nu(x)| \leq C_s \bigg[\bigg(\frac{1}{\mu(B(x,\varepsilon'/5))} \int_{B(x,\varepsilon'/5)} |T_\delta \nu_1|^s \, d\mu \bigg)^{1/s} \\ + \bigg(\frac{1}{\mu(B(x,\varepsilon'/5))} \int_{B(x,\varepsilon'/5)} |T_\delta \nu|^s \, d\mu \bigg)^{1/s} + M_\mu \nu(x) \bigg]. \end{aligned} \tag{25}$$

By (23), the second term on the right-hand side of (25) can be estimated by $\widetilde{M}_\mu(|T_\delta \nu|^s)(x)^{1/s}$. On the other hand, the first term is estimated using Kolmogorov's inequality, the boundedness of T_δ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, and (23):

$$\begin{aligned} \bigg(\frac{1}{\mu(B(x,\varepsilon'/5))} \int_{B(x,\varepsilon'/5)} |T_\delta \nu_1|^s \, d\mu \bigg)^{1/s} &\lesssim \frac{\|T_\delta \nu_1\|_{L^{1,\infty}(\mu)}}{\mu(B(x,\varepsilon'/5))} \\ &\lesssim \frac{\|\nu_1\|}{\mu(B(x,\varepsilon'/5))} \\ &\lesssim M_\mu \nu(x). \end{aligned}$$

Now (22) follows. □

As in the classical doubling case, a direct consequence of Cotlar's inequality and Theorem 6 is the following result.

Theorem 10. *Let μ be a Radon measure on $\mathbb{R}^d$ of degree n. If T_μ is an n-dimensional CZO bounded in $L^2(\mu)$, then $T_{\mu,*}$ is bounded in $L^p(\mu)$, $p \in (1,\infty)$, and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$.*

Proof. By Theorem 6, interpolation, and duality, T_μ is bounded in $L^p(\mu)$, $p \in (1,\infty)$, and from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. Then, by Cotlar's inequality it is clear that $T_{*,\delta}$ is bounded in $L^p(\mu)$, $p \in (1,\infty)$, uniformly on $\delta > 0$. Hence, by monotone convergence, T_* is also bounded in $L^p(\mu)$, $p \in (1,\infty)$. The boundedness of T_* from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$, as in the classical doubling case, requires some additional work. By monotone convergence, it is enough to show that

$$\mu\{x : T_{*,\delta}\nu(x) > \lambda\} \leq \frac{\|\nu\|}{\lambda}. \tag{26}$$

By Cotlar's inequality (19) for $s = 1/2$, we have for $\lambda > 0$

$$\begin{aligned} \mu\{x : T_{*,\delta}\nu(x) > \lambda\} \leq \mu\Big\{x : M_\mu \nu(x) > \frac{\lambda}{2C_{1/2}}\Big\} \\ + \mu\Big\{x : \big(\widetilde{M}_\mu(|T_\delta \nu|^{1/2})(x)\big)^2 > \frac{\lambda}{2C_{1/2}}\Big\}. \end{aligned} \tag{27}$$

The first term on the right-hand side of (27) satisfies

$$\mu\Big\{x : M_\mu \nu(x) > \frac{\lambda}{2C_{1/2}}\Big\} \le C\frac{\|\nu\|}{\lambda},$$

by the boundedness of M_μ from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. To estimate the second term on the right-hand side of (27) we introduce the non-centered restricted maximal Hardy-Littlewood operator N_μ introduced above. Obviously,

$$\begin{aligned}\mu\Big\{x : \big(\widetilde{M}_\mu(|T_\delta \nu|^{1/2})(x)\big)^2 > \frac{\lambda}{2C_{1/2}}\Big\}& \\ \le \mu\Big\{x : N_\mu(|T_\delta \nu|^{1/2})(x) > \frac{\lambda^{1/2}}{(2C_{1/2})^{1/2}}\Big\}&.\end{aligned} \tag{28}$$

Recall that, by the doubling condition on the balls $B(y, r)$, the operator N_μ is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. In fact, for any $\sigma \in M(\mathbb{R}^d)$, the following sharper condition holds:

$$\mu\{x : N_\mu \sigma(x) > \lambda\} \le C\frac{1}{\lambda}\int_{\{x:N_\mu\sigma(x)>\lambda\}} d|\sigma|. \tag{29}$$

This inequality follows from the usual arguments, using the $5r$-covering theorem, as in [Ma2, pp. 40–42]. Notice that an estimate such as (29) does not hold for the centered operator $\widetilde{M}_\mu$, in general.

Now, if we denote

$$A = \Big\{x : N_\mu(|T_\delta \nu|^{1/2})(x) > \frac{\lambda^{1/2}}{(2C_{1/2})^{1/2}}\Big\},$$

applying (29) to $g = |T_\delta \nu|^{1/2}$ and by Kolmogorov's inequality, we obtain

$$\begin{aligned}\mu(A) &\le C\frac{1}{\lambda^{1/2}}\int_A |T_\delta \nu|^{1/2}\, d\mu \\ &\le C\frac{1}{\lambda^{1/2}}\mu(A)^{1/2}\|T_\delta \nu\|^{1/2}_{L^{1,\infty}(\mu)} \\ &\le C\frac{1}{\lambda^{1/2}}\mu(A)^{1/2}\|\nu\|^{1/2}.\end{aligned}$$

Therefore,

$$\mu(A) \le C\frac{\|\nu\|}{\lambda},$$

and by (28),

$$\mu\Big\{x : \big(\widetilde{M}_\mu(|T_\delta \nu|^{1/2})(x)\big)^2 > \frac{\lambda}{2C_{1/2}}\Big\} \le C\frac{\|\nu\|}{\lambda},$$

and so (26) holds. □

6. The $T(1)$ theorem, $RBMO$, and H^1

Let us introduce some notation and definitions. Given $\rho > 1$, we say that $f \in L^1_{\text{loc}}(\mu)$ belongs to the space $BMO_\rho(\mu)$ if

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f - m_Q(f)|\, d\mu < \infty,$$

where the supremum is taken over all the squares in $\mathbb{R}^d$ and $m_Q(f)$ is the μ-mean of f over Q.

A Calderón-Zygmund operator T_μ is said to be weakly bounded if

$$\bigl|\langle T_{\mu,\varepsilon}\chi_Q, \chi_Q\rangle\bigr| \leq C\mu(Q) \quad \text{for all the cubes } Q \subset \mathbb{R}^d, \text{ uniformly on } \varepsilon > 0.$$

Notice that if T_μ is antisymmetric, then the left-hand side above equals zero and so T_μ is weakly bounded.

Now we are ready to state the $T(1)$ theorem:

Theorem 11. *Let μ be a Radon measure on $\mathbb{R}^d$ of degree n, and let T be an n-dimensional Calderón-Zygmund operator. The following conditions are equivalent:*

(a) *T_μ is bounded on $L^2(\mu)$.*
(b) *T_μ is weakly bounded and, for some $\rho > 1$, we have that $T_{\mu,\varepsilon}(1)$, $T^*_{\mu,\varepsilon}(1) \in BMO_\rho(\mu)$ uniformly on $\varepsilon > 0$.*
(c) *There exists some constant C_5 such that for all $\varepsilon > 0$ and all the cubes $Q \subset \mathbb{R}^d$,*

$$\|T_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \leq C_5\mu(Q)^{1/2} \quad \textit{and} \quad \|T^*_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \leq C_5\mu(Q)^{1/2}.$$

The classical way of stating the $T(1)$ theorem is the equivalence (a) $\Leftrightarrow$ (b). However, for some applications it is sometimes more practical to state the result in terms of the L^2 boundedness of T_μ and T^*_μ over characteristic functions of cubes, i.e. (a) $\Leftrightarrow$ (c).

Theorem 11 is the extension of the classical $T(1)$ theorem of David and Journé to measures of degree n which may be non doubling. The result was proved by Nazarov, Treil and Volberg in [NTV1], although not exactly in the form stated above. An independent proof for the particular case of the Cauchy transform was obtained simultaneously in [To1].

Let us remark that the boundedness of T_μ on $L^2(\mu)$ does not imply the boundedness of T_μ from $L^\infty(\mu)$ into $BMO(\mu)$ (this is the space $BMO_\rho(\mu)$ with parameter $\rho = 1$), and $T_{\mu,\varepsilon}(1), T^*_{\mu,\varepsilon}(1) \notin BMO(\mu)$ uniformly on $\varepsilon > 0$, in general. On the contrary, one can show that if T_μ is bounded on $L^2(\mu)$, then it is also bounded from $L^\infty(\mu)$ into $BMO_\rho(\mu)$, for $\rho > 1$, by arguments similar to the classical ones for homogeneous spaces. However, the space $BMO_\rho(\mu)$ has some drawbacks. For example, it depends on the parameter ρ and it does not satisfy the John-Nirenberg inequality. To solve these problems, in [To3] a new space called $RBMO(\mu)$ has been introduced. The precise definition is the following: we say that $f \in RBMO(\mu)$ if it belongs to $BMO_2(\mu)$ (i.e. satisfies (30) with $\rho = 2$), and moreover, for all $(2, 2^{d+1})$-doubling cubes $Q \subset R$,

$$|m_Q f - m_R f| \leq C_f K_{Q,R},$$

where

$$K_{Q,R} = 1 + \int_{2R\setminus Q} \frac{1}{|x - x_Q|^n} \, d\mu(x).$$

Let us remark that the definition of $RBMO(\mu)$ does not depend on the choice of the parameter $\rho = 2$. Moreover, $RBMO(\mu)$ is a subspace of $BMO_\rho(\mu)$ for all $\rho > 1$, and it coincides with $BMO(\mu)$ when μ is an AD-regular measure. Further, $RBMO(\mu)$ satisfies a John-Nirenberg type inequality, and all CZO's which are bounded on $L^2(\mu)$ are also bounded from $L^\infty(\mu)$ into $RBMO(\mu)$. For these reasons $RBMO(\mu)$ seems to be a good substitute of the classical space BMO for non doubling measures of degree n.

One can also show that the statement (b) in Theorem 11 is equivalent to

(b') T_μ is weakly bounded and we have that $T_{\mu,\varepsilon}(1) T^*_{\mu,\varepsilon}(1) \in RBMO(\mu)$ uniformly on $\varepsilon > 0$.

See [To3] for all the details.

The predual of $RBMO(\mu)$ is a Hardy type space, which we proceed to define. For a fixed $\rho > 1$, a function $b \in L^1_{loc}(\mu)$ is called an *atomic block* if

1. there exists some cube R such that $\mathrm{supp}(b) \subset R$,
2. $\displaystyle\int b \, d\mu = 0$,
3. there are functions a_1, a_2 supported on cubes $Q_1, Q_2 \subset R$ and numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^\infty(\mu)} \leq \big(\mu(\rho Q_j) K_{Q_j,R}\big)^{-1} \quad \text{for } j = 1, 2.$$

We denote

$$|b|_{H^1(\mu)} = |\lambda_1| + |\lambda_2|$$

(to be rigorous, we should think that b is not only a function, but a 'structure' formed by the function b, the cubes R and Q_j, the functions a_j, etc.). Then, we say that $f \in H^1(\mu)$ if there are atomic blocks b_i such that

$$f = \sum_{i=1}^{\infty} b_i, \tag{31}$$

with $\sum_i |b_i|_{H^1(\mu)} < \infty$ (notice that this implies that the sum in (31) converges in $L^1(\mu)$). The $H^1(\mu)$ norm of f is

$$\|f\|_{H^1(\mu)} = \inf \sum_i |b_i|_{H^1(\mu)},$$

where the infimum is taken over all the possible decompositions of f in atomic blocks.

One can check that the definition of $H^1(\mu)$ does not depend on the constant $\rho > 1$. The $H^1(\mu)$ norms for different choices of $\rho > 1$ are equivalent. As mentioned above,

$$(H^1(\mu))^* = RBMO(\mu).$$

Moreover, it is not difficult to show that if an n-dimensional Calderón-Zygmund operator is bounded in $L^2(\mu)$, then it is also bounded from $H^1(\mu)$ into $L^1(\mu)$. See [To3] for the detailed arguments and other related results, such as an interpolation theorem between the pairs $(L^\infty(\mu), RBMO(\mu))$ and $(H^1(\mu), L^1(\mu))$. Let us also remark that the space $H^1(\mu)$ admits a characterization in terms of a grand maximal function. See [To5].

Many more results on Calderón-Zygmund theory with non doubling measures have been proved recently. For example, several $T(b)$ type theorems have been obtained in [DM], [Dd4], [NTV3], [NTV4], [NTV5]. There are also results concerning weights [GCM1], [HY], [MM], [OP], [To13]; commutators [CS], [HMY3], [MY], [To3]; multilinear commutators [HMY1], [HMY2]; fractional integrals [GCM2], [GCG1]; Lipschitz spaces [GCG2]; maximal singular integrals [HMY4], and more on Hardy spaces [CMY]; etc.

7. The Cauchy transform

Recall that the Cauchy transform is the CZO on $\mathbb{C}$ originated by the kernel

$$k(x, y) := \frac{1}{y - x}, \quad x, y \in \mathbb{C}.$$

It is denoted by $\mathcal{C}$. That is to say,

$$\mathcal{C}\mu(x) := \int \frac{1}{y-x}\, d\mu(y), \quad x \in \mathbb{C} \setminus \operatorname{supp}(\mu).$$

As is well know, the Cauchy transform plays a fundamental role in complex analysis, because of Cauchy reproducing formula. When μ is fixed, we set $\mathcal{C}_\mu(f) = \mathcal{C}(f\, d\mu)$.

Of course, the results and techniques of Calderón-Zygmund theory explained in the preceding sections apply to the Cauchy transform (with the parameters $n = 1$, $d = 2$). Moreover, because of the relationship of the Cauchy kernel with Menger curvature, discovered by Melnikov [Me], some results (such as the $T(1)$ theorem) are easier to prove for the Cauchy transform than for general CZO's.

In this section we will describe in detail the relationship between the Cauchy transform and Menger curvature, and using this, we will give a rather simple proof of the $T(1)$ theorem for the Cauchy singular integral operator, i.e. for $\mathcal{C}_\mu$.

7.1. The curvature of a measure. Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $R(x,y,z) = \infty$, $c(x,y,z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we let $c(x,y,z) = 0$. For a positive Radon measure μ, we set

$$c_\mu^2(x) = \iint c^2(x,y,z)\, d\mu(y)\, d\mu(z),$$

and we define the *curvature of* μ as

$$c^2(\mu) = \int c_\mu^2(x)\, d\mu(x) = \iiint c^2(x,y,z)\, d\mu(x)\, d\mu(y)\, d\mu(z). \tag{32}$$

The notion of curvature of measures was introduced by Melnikov [Me] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible of the big recent advances in connection with analytic capacity. The notion of curvature is connected to the Cauchy transform by the following result, proved by Melnikov and Verdera.

Proposition 12. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. We have*

$$\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} = \frac{1}{6}c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})), \tag{33}$$

where $c^2_\varepsilon(\mu)$ is the ε-truncated version of $c^2(\mu)$ (defined as in the right-hand side of (32)*, but with the triple integral over $\{x, y, z \in \mathbb{C} : |x-y|, |y-z|, |x-z| > \varepsilon\}$), and $|O(\mu(\mathbb{C}))| \leq C\mu(\mathbb{C})$.*

The identity (33) is remarkable because it relates an analytic notion (the Cauchy transform of a measure) with a metric-geometric one (curvature). It played a key role in many of the recent results on analytic capacity.

Proof. We have

$$\begin{aligned}
\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} &= \int\left|\int_{|y-x|>\varepsilon} \frac{1}{y-x}\,d\mu(y)\right|^2 d\mu(x) \\
&= \iiint_{\substack{|x-y|>\varepsilon\\ |x-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\,d\mu(y)\,d\mu(z)\,d\mu(x) \\
&= \iiint_{\substack{|x-y|>\varepsilon\\ |x-z|>\varepsilon\\ |y-z|>\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\,d\mu(y)\,d\mu(z)\,d\mu(x) \\
&\quad + \iiint_{\substack{|x-y|>\varepsilon\\ |x-z|>\varepsilon\\ |y-z|\leq\varepsilon}} \frac{1}{(y-x)\overline{(z-x)}}\,d\mu(y)\,d\mu(z)\,d\mu(x) \\
&=: I_1 + I_2.
\end{aligned} \tag{34}$$

Consider first the integral I_1. By Fubini, permuting x, y, z, we get,

$$I_1 = \frac{1}{6}\iiint \sum_{s\in S_3} \frac{1}{(z_{s_2}-z_{s_1})\overline{(z_{s_3}-z_{s_1})}}\,d\mu(z_1)\,d\mu(z_2)\,d\mu(z_3),$$

where S_3 is the group of permutations of three elements. An elementary calculation shows that

$$\sum_{s\in S_3} \frac{1}{(z_{s_2}-z_{s_1})\overline{(z_{s_3}-z_{s_1})}} = c^2(z_1, z_2, z_3).$$

So we get

$$I_1 = \frac{1}{6}c^2_\varepsilon(\mu).$$

To estimate the integral I_2 in (34), notice that, by the conditions in the domain of integration, the side with vertices y, z is the shortest in the triangle formed by the vertices x, y, z. Thus, $|x-y| \approx |x-z|$, and so

$$|I_2| \lesssim \iiint_{\substack{|x-y|>\varepsilon \\ |x-z|>\varepsilon \\ |y-z|\le\varepsilon}} \frac{1}{|y-x|^2}\, d\mu(y)\, d\mu(z)\, d\mu(x).$$

Integrating with respect to z, by the linear growth of μ, we derive

$$|I_2| \lesssim \varepsilon \iint_{|x-y|>\varepsilon} \frac{1}{|y-x|^2}\, d\mu(y)\, d\mu(x).$$

Using again the linear growth of μ, splitting the domain $\{y : |x-y| > \varepsilon\}$ into annuli, it is easy to check that

$$\int_{y:|x-y|>\varepsilon} \frac{1}{|y-x|^2}\, d\mu(y) \lesssim \frac{1}{\varepsilon}.$$

Thus we get,

$$|I_2| \lesssim \mu(\mathbb{C}),$$

and the proposition follows. □

Lemma 13. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth and let $f \in L^2(\mu)$ be a non negative real function. Then we have*

$$\begin{aligned} 4 \int |\mathcal{C}_\varepsilon(f\, d\mu)|^2\, d\mu = &\iiint_{\substack{|x-y|>\varepsilon \\ |x-z|>\varepsilon \\ |y-z|>\varepsilon}} c^2(x,y,z) f(x) f(y)\, d\mu(x)\, d\mu(y)\, d\mu(z) \\ &- 2\,\mathrm{Re} \int (\mathcal{C}_\varepsilon \mu) \overline{\mathcal{C}_\varepsilon(f\, d\mu)} f\, d\mu + O(\|f\|^2_{L^2(\mu)}). \end{aligned}$$

The proof is similar to the one of Proposition 12. We leave it as an exercise for the reader. Otherwise, see [Ve2, Lemma 1] for the details, for example.

7.2. The $T(1)$ theorem for the Cauchy singular integral operator. We will prove the following version of the $T(1)$ theorem for the Cauchy singular integral operator:

Theorem 14. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. The following conditions are equivalent:*

(a) *$\mathcal{C}_\mu$ is bounded on $L^2(\mu)$.*
(b) *For all $\varepsilon > 0$ and all the squares $Q \subset \mathbb{C}$,*

$$\|\mathcal{C}_{\mu,\varepsilon}\chi_Q\|_{L^2(\mu|Q)} \leq C\mu(Q)^{1/2},$$

with C independent of ε.
(c) *For all the squares $Q \subset \mathbb{C}$,*

$$c^2(\mu_{|Q}) \leq C\mu(Q).$$

To prove this theorem we will exploit the relationship between the Cauchy transform and Menger curvature, and we will use a good λ inequality, as in the proofs of the same result in [To1] and [Ve2]. The proof below is new although it has some resemblaces with the ones in the references just mentioned, especially with [Ve2].

Notice that the equivalence (b) $\Leftrightarrow$ (c) follows by a direct application of (33) to the measure $\mu_{|Q}$, for all the squares $Q \subset \mathbb{C}$. So we only have to prove (b) $\Rightarrow$ (a). To this end, we need the following key lemma:

Lemma 15. *Let μ be a finite measure with linear growth on $\mathbb{C}$, that is,*

$$\mu(B(x,r)) \leq C_0 r \quad \textit{for all } x \in \mathbb{C}, r > 0.$$

Suppose that $\|\mathcal{C}_\varepsilon \mu\|^2_{L^2(\mu)} \leq C_1\mu(\mathbb{C})$ for all $\varepsilon > 0$. Then there exists a subset $G \subset \mathbb{C}$ with $\mu(G) \geq \mu(\mathbb{C})/4$ such that $\mathcal{C}_{\mu_{|G}} : L^2(\mu_{|G}) \to L^2(\mu_{|G})$ is bounded with norm bounded above by some constant depending only on C_0 and C_1.

Proof. From the assumptions in the lemma and Proposition 13, we deduce

$$c^2(\mu) \leq C_6\mu(\mathbb{C}).$$

Given $C_7 > 0$, let

$$A_\varepsilon := \big\{x \in \mathbb{C} : |\mathcal{C}_\varepsilon\mu(x)| \leq C_7 \text{ and } c^2_\mu(x) \leq C_7^2\big\}.$$

Since $\int c^2_\mu(x)\,d\mu(x) = c^2(\mu) \leq C_6\mu(\mathbb{C})$ and $\int |\mathcal{C}_\varepsilon\mu|^2\,d\mu \leq C\mu(\mathbb{C})$, we infer that $\mu(A_\varepsilon) \geq \mu(\mathbb{C})/2$ if C_7 is chosen big enough, by Chebyshev.

We want to show that the Cauchy integral operator $\mathcal{C}_{\mu_{|A_\varepsilon},\varepsilon}$ is bounded on $L^2(\mu_{|A_\varepsilon})$. To this end we introduce an auxiliary "curvature operator": for

$x, y \in A_\varepsilon$, consider the kernel $k(x,y) := \int c^2(x,y,z)\,d\mu(z)$, and let T be the operator

$$Tf(x) = \int k(x,y)f(y)\,d\mu(y).$$

By Schur's lemma, T is bounded on $L^p(\mu_{|A_\varepsilon})$ for all $p \in [1,\infty]$, because for all $x \in A_\varepsilon$,

$$\begin{aligned}\int k(x,y)\,d\mu_{|A_\varepsilon}(y) &= \int k(y,x)\,d\mu_{|A_\varepsilon}(y)\\ &= \int_{y\in A_\varepsilon} c^2(x,y,z)\,d\mu(y)\,d\mu(z) \le c_\mu^2(x) \le C_7^2.\end{aligned}$$

Recall that given a non negative (real) function f supported on A_ε, by Lemma 13 we have

$$\begin{aligned}4\int |\mathcal{C}_\varepsilon(f\,d\mu)|^2\,d\mu = &\iiint_{\substack{|x-y|>\varepsilon\\|x-z|>\varepsilon\\|y-z|>\varepsilon}} c^2(x,y,z)f(x)f(y)\,d\mu(x)\,d\mu(y)\,d\mu(z)\\ &- 2\,\mathrm{Re}\int(\mathcal{C}_\varepsilon\mu)\overline{\mathcal{C}_\varepsilon(f\,d\mu)}f\,d\mu + O(\|f\|^2_{L^2(\mu)}).\end{aligned}$$

Thus,

$$\int |\mathcal{C}_\varepsilon(f\,d\mu)|^2\,d\mu \le \frac{1}{4}\big|\langle Tf,f\rangle\big| + \frac{1}{2}\int\big|(\mathcal{C}_\varepsilon\mu)\mathcal{C}_\varepsilon(f\,d\mu)f\big|\,d\mu + C\|f\|^2_{L^2(\mu)}. \quad (35)$$

To estimate the first term on the right side we use the $L^2(\mu_{|A_\varepsilon})$ boundedness of T (recall that $\mathrm{supp}(f) \subset A_\varepsilon$):

$$\big|\langle Tf,f\rangle\big| \le \|Tf\|_{L^2(\mu)}\|f\|_{L^2(\mu)} \le C\|f\|^2_{L^2(\mu)}.$$

To deal with the second integral on the right side of (35), notice that $|\mathcal{C}_\varepsilon\mu| \le C_7$ on the support of f, and so

$$\int\big|(\mathcal{C}_\varepsilon\mu)\mathcal{C}_\varepsilon(f\,d\mu)f\big|\,d\mu \le C_7\int\big|\mathcal{C}_\varepsilon(f\,d\mu)f\big|\,d\mu \le C_7\|\mathcal{C}_\varepsilon(f\,d\mu)\|_{L^2(\mu)}\|f\|_{L^2(\mu)}.$$

By (35) we get

$$\|\mathcal{C}_\varepsilon(f\,d\mu)\|^2_{L^2(\mu)} \le C\|f\|^2_{L^2(\mu)} + \frac{C_7}{2}\|\mathcal{C}_\varepsilon(f\,d\mu)\|_{L^2(\mu)}\|f\|_{L^2(\mu)},$$

which implies that $\|\mathcal{C}_\varepsilon(f\,d\mu)\|_{L^2(\mu)} \le C\|f\|_{L^2(\mu)}$.

So far we have proved the $L^2(\mu_{|A_\varepsilon})$ boundedness of $\mathcal{C}_{\mu_{|A_\varepsilon},\varepsilon}$. If A_ε were independent of ε, we would set $\nu := \mu_{|A_\varepsilon}$ and we would be done. Unfortunately this is not the case and we have to work a little more. We set

$$G_\varepsilon := \bigl\{x \in \mathbb{C} : |\mathcal{C}_{\varepsilon,*}\mu(x)| \le C_8 \text{ and } c_\mu^2(x) \le C_8^2\bigr\},$$

where C_8 is some constant big enough (with $C_8 > C_7$) to be chosen below. By Theorem 10 and the discussion above, we know that $\mathcal{C}_{\varepsilon,*}$ is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu_{|A_\varepsilon})$ (with constants independent of ε). Thus,

$$\mu\bigl\{x \in A_\varepsilon : |\mathcal{C}_{\varepsilon,*}\mu(x)| > C_8\bigr\} \le \frac{C\mu(\mathbb{C})}{C_8}.$$

If C_8 is big enough, the right-hand side of the preceding inequality is $\le \mu(\mathbb{C})/4 \le \mu(A_\varepsilon)/2$. Thus, $\mu(G_\varepsilon) \ge \mu(\mathbb{C})/4$.

We set

$$G := \bigcap_{\varepsilon>0} G_\varepsilon.$$

Notice that, by definition, $G_\varepsilon \subset G_\delta$ if $\varepsilon > \delta$ and so we have

$$\mu(G) = \lim_{\varepsilon\to 0} \mu(G_\varepsilon) \ge \frac{1}{4}\mu(\mathbb{C}).$$

By the same argument used for A_ε, it follows that $\mathcal{C}_{\mu_{|G_\varepsilon},\varepsilon}$ is bounded on $L^2(\mu_{|G_\varepsilon})$ (with constant independent of ε), and thus $\mathcal{C}_{\mu_{|G}}$ is bounded on $L^2(\mu_{|G})$. □

Proof of Theorem 14. As remarked above, we only have to prove (b) ⇒ (a). We will use a *good λ inequality*: we will show that there exist some absolute constant $\eta > 0$ such that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{aligned}\mu\bigl\{x : \mathcal{C}_{\mu,*}f(x) > (1+\varepsilon)\lambda,\ M_\mu f(x) \le \delta\lambda\bigr\}& \\ \le (1-\eta)\mu\bigl\{x : \mathcal{C}_{\mu,*}f(x) > \lambda\bigr\}&\end{aligned} \tag{36}$$

for $f \in L^1(\mu)$ (with compact support, say). It is easy to check that this implies that $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$, by standard arguments.

To prove (36), consider a Whitney decomposition of the open set

$$\Omega_\lambda = \bigl\{x : \mathcal{C}_{\mu,*}f(x) > \lambda\bigr\}$$

into a family of dyadic squares $\{Q_i\}_{i\in I}$ with disjoint interior such that $5Q_i \subset \Omega_\lambda$ and $20Q_i \cap \Omega_\lambda^c \neq \emptyset$ for each $i \in I$, and so that, moreover,

$$\sum_{i\in I} \chi_{5Q_i} \leq C_9 \chi_{\Omega_\lambda}. \tag{37}$$

That is, the family $\{5Q_i\}_{i\in I}$ has bounded overlap.

Consider a square Q_i, $i \in I$. We claim that if $x \in Q_i$ satisfies $\mathcal{C}_{\mu,*}f(x) > (1+\varepsilon)\lambda$ and $M_\mu f(x) \leq \delta\lambda$, then

$$\mathcal{C}_{\mu,*}(\chi_{2Q_i} f)(x) > \varepsilon\lambda/2, \tag{38}$$

assuming that δ is small enough. To show that this holds, take $z \in 20Q_i \setminus \Omega_\lambda$, so that $\mathcal{C}_{\mu,*}f(z) \leq \lambda$. Since $x, z \in 20Q_i$, using the linear growth of μ, by standard arguments it follows easily that

$$\left|\mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 40Q_i} f)(x) - \mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 40Q_i} f)(z)\right| \leq C M_\mu f(x).$$

By straightforward estimates, since $x \in Q_i$, it is also easy to check that $\mathcal{C}_{\mu,*}(\chi_{40Q_i\setminus 2Q_i} f)(x) \leq C M_\mu f(x)$. Therefore,

$$\begin{aligned}
\mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 2Q_i} f)(x) &\leq \mathcal{C}_{\mu,*}(\chi_{40Q_i\setminus 2Q_i} f)(x) + \mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 40Q_i} f)(x) \\
&\leq \mathcal{C}_{\mu,*}(\chi_{40Q_i\setminus 2Q_i} f)(x) + \mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 40Q_i} f)(z) \\
&\quad + \left|\mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 40Q_i} f)(x) - \mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 40Q_i} f)(z)\right| \\
&\leq C M_\mu f(x) + \lambda + C M_\mu f(x) \leq (1 + C_{10}\delta)\lambda,
\end{aligned}$$

and thus

$$\mathcal{C}_{\mu,*}(\chi_{2Q_i} f)(x) \geq \mathcal{C}_{\mu,*}f(x) - \mathcal{C}_{\mu,*}(\chi_{\mathbb{C}\setminus 2Q_i} f)(x) > (1+\varepsilon)\lambda - (1 + C_{10}\delta)\lambda.$$

So the claim follows if we choose $C_{10}\delta \leq \varepsilon/2$.

Now, since for all $\varepsilon > 0$ we have $\|\mathcal{C}_{\mu,\varepsilon}(\chi_{Q_i})\|^2_{L^2(\mu_{|Q_i})} \leq C\mu(Q_i)$, by Lemma 15 applied to the measure $\mu_{|Q_i}$, there exists some subset $G_i \subset Q_i$ with $\mu(G_i) \geq \mu(Q_i)/4$ such that the Cauchy transform is bounded in $L^2(\mu_{|G_i})$. By Theorem 11, we infer that $\mathcal{C}_{\mu,*}$ is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu_{|G_i})$. Together with (38) this implies that

$$\begin{aligned}
&\mu\{x \in G_i : \mathcal{C}_{\mu,*}f(x) > (1+\varepsilon)\lambda,\ M_\mu f(x) \leq \delta\lambda\} \\
&\qquad\qquad \leq \mu\{x \in G_i : \mathcal{C}_{\mu,*}(\chi_{2Q_i} f)(x) > \varepsilon\lambda/2\} \\
&\qquad\qquad \leq \frac{C}{\varepsilon\lambda}\|\chi_{2Q_i} f\|_{L^1(\mu)}.
\end{aligned}$$

Notice that, if Q_i contains some point x such that $M_\mu f(x) \leq \delta\lambda$, then

$$\begin{aligned}\|\chi_{2Q_i} f\|_{L^1(\mu)} &\leq \int_{B(x,2\ell(Q_i))} |f|\, d\mu \\ &\leq \mu(B(x,2\ell(Q_i)))M_\mu f(x) \\ &\leq \mu(5Q_i)M_\mu f(x) \\ &\leq \delta\lambda\mu(5Q_i).\end{aligned}$$

Thus,

$$\mu\{x \in G_i : \mathcal{C}_{\mu,*}f(x) > (1+\varepsilon)\lambda, M_\mu f(x) \leq \delta\lambda\} \leq \frac{C_{11}\delta}{\varepsilon}\mu(5Q_i).$$

Using the preceding estimate for all the squares Q_i, $i \in I$, and the fact that $\mu(G_i) \geq \mu(Q_i)/4$, we get

$$\begin{aligned}&\mu\big\{x : \mathcal{C}_{\mu,*}f(x) > (1+\varepsilon)\lambda, M_\mu f(x) \leq \delta\lambda\big\} \\ &\quad \leq \sum_{i\in I}\mu(Q_i \setminus G_i) + \sum_{i\in I}\mu\big\{x \in G_i : \mathcal{C}_{\mu,*}f(x) > (1+\varepsilon)\lambda,\ M_\mu f(x) \leq \delta\lambda\big\} \\ &\quad \leq \sum_{i\in I}\frac{3}{4}\mu(Q_i) + \sum_{i\in I}\frac{C_{11}\delta}{\varepsilon}\mu(5Q_i)\end{aligned}$$

By the bounded overlap (37), we obtain

$$\mu\big\{x : \mathcal{C}_{\mu,*}f(x) > (1+\varepsilon)\lambda, M_\mu f(x) \leq \delta\lambda\big\} \leq \frac{3}{4}\mu(\Omega_\lambda) + \frac{C_9 C_{11}\delta}{\varepsilon}\mu(\Omega_\lambda).$$

Therefore, if we choose $\delta \leq C_9^{-1}C_{11}^{-1}\varepsilon/8$, the right-hand side is bounded above by $\frac{7}{8}\mu(\Omega_\lambda)$, and so (36) follows with $\eta = 1/8$. We are done. □

8. Analytic capacity

8.1. Definition. The *analytic capacity* of a compact set $E \subset \mathbb{C}$ is

$$\gamma(E) := \sup|f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

The notion of analytic capacity was introduced by Ahlfors [Ah] in the 1940's in order to study the removability of singularities of bounded analytic functions. A compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if for any open set Ω containing E, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. AHLFORS showed that E is removable if and only if $\gamma(E) = 0$.

Painlevé's problem consists of characterizing removable singularities for bounded analytic functions in a metric/geometric way. By AHLFORS' result this is equivalent to describe compact sets with positive analytic capacity in metric/geometric terms.

VITUSHKIN in the 1950's and 1960's showed that analytic capacity plays a central role in problems of uniform rational approximation on compact sets of the complex plane. Many results obtained by VITUSHKIN in connection with uniform rational approximation are stated in terms of γ. See [Vi2], or [Ve1] for a more modern approach, for example. Further, because its applications to this type of problems he raised the question of the semiadditivity of γ. Namely, does there exist an absolute constant C such that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F))\,?$$

This was shown to be true in [To8].

Proving the semiadditivity of analytic capacity is out of the scope of these lecture notes. However, below we will show that the so called capacity γ_+ is semiadditive. The semidditivity of the analytic capacity γ follows then from the comparability of γ and γ_+, as shown in [To8]. As we will see, the Calderón-Zygmund theory with non doubling measures that we have developed in the previous sections will play a fundamental role.

8.2. Basic properties of analytic capacity. One should keep in mind that, in a sense, analytic capacity measures the size of a set as a non removable singularity for bounded analytic functions. A direct consequence of the definition is that

$$E \subset F \quad \Longrightarrow \quad \gamma(E) \leq \gamma(F).$$

Moreover, it is also easy to check that analytic capacity is translation invariant:

$$\gamma(z + E) = \gamma(E) \quad \text{for all } z \in \mathbb{C}.$$

Concerning dilations, we have

$$\gamma(\lambda E) = |\lambda| \gamma(E) \quad \text{for all } \lambda \in \mathbb{C}.$$

Further, if E is connected, then

$$\operatorname{diam}(E)/4 \leq \gamma(E) \leq \operatorname{diam}(E).$$

The second inequality follows from the fact that the analytic capacity of a closed disk coincides with its radius, and the first one is a consequence of Koebe's 1/4 theorem (see [Ga, Chapter VIII] for the details, for example).

8.3. Relationship with Hausdorff measure. The relationship between Hausdorff measure and analytic capacity is the following:

- If $\dim_H(E) > 1$ (here $\dim_H$ stands for the Hausdorff dimension), then $\gamma(E) > 0$. This result follows easily from Frostman's Lemma.
- $\gamma(E) \leq \mathcal{H}^1(E)$, where $\mathcal{H}^1$ is the one dimensional Hausdorff measure, or length. This follows from Cauchy's integral formula, and it was proved by PAINLEVÉ about one hundred years ago. Observe that, in particular we deduce that if $\dim_H(E) < 1$, then $\gamma(E) = 0$.

By the statements above, it turns out that dimension 1 is the critical dimension in connection with analytic capacity. Moreover, a natural question arises: is it true that $\gamma(E) > 0$ if and only if $\mathcal{H}^1(E) > 0$?

VITUSHKIN showed that the answer is *no*. Indeed, he constructed a compact set in $\mathbb{C}$ with positive length and vanishing analytic capacity. This set was purely unrectifiable. That is, it intersects any rectifiable curve at most in a set of zero length. Motivated by this example (and others, I guess) VITUSHKIN conjectured that pure unrectifiability is a necessary and sufficient condition for vanishing analytic capacity, for sets with finite length.

GUY DAVID [Dd4] showed in 1998 that Vitushkin's conjecture is true:

Theorem 16. *Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then, $\gamma(E) = 0$ if and only if E is purely unrectifiable.*

Let us remark that the "if" part of the theorem is not due to DAVID (it follows from CALDERÓN's theorem on the L^2 boundedness of the Cauchy transform on Lipschitz graphs). The "only if" part of the theorem, which is more difficult, is the one proved by David. See also [MMV], [DM] and [Lé] for some preliminary contributions to the proof.

Theorem 16 is the solution of Painlevé's problem for sets with finite length. The analogous result is false for sets with infinite length. For this type of sets there is no such a nice geometric solution of Painlevé's problem, and we have to content ourselves with a characterization such as the one in Corollary 23 below (at least, for the moment).

8.4. The capacity γ_+ and the Cauchy transform. The capacity γ_+ of a compact set $E \subset \mathbb{C}$ is

$$\gamma_+(E) := \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E,\ \|\mathbb{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1\}. \tag{40}$$

That is, γ_+ is defined as γ in (39) with the additional constraint that f should coincide with $\mathcal{C}\mu$, where μ is some positive Radon measure supported on E (observe that $(\mathcal{C}\mu)'(\infty) = -\mu(\mathbb{C})$ for any Radon measure μ). To be precise, there is another little difference: in (39) we asked $\|f\|_{L^\infty(\mathbb{C}\setminus E)} \leq 1$, while in (40) $\|f\|_{L^\infty(\mathbb{C})} \leq 1$ (for $f = \mathbb{C}\mu$). Trivially, we have $\gamma_+(E) \leq \gamma(E)$.

The following lemma relates weak $(1,1)$ estimates for the Cauchy integral operator with L^∞ estimates (which in its turn are connected with γ_+ and γ).

Lemma 17. *Let μ be a Radon measure with linear growth on $\mathbb{C}$. The following statements are equivalent:*

(a) *The Cauchy transform is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$.*
(b) *For any set $A \subset \mathbb{C}$ there exists some function h supported on A, with $0 \leq h \leq 1$, such that $\int h\, d\mu \geq C^{-1}\mu(A)$ and $\|\mathcal{C}_\varepsilon(h\, d\mu)\|_{L^\infty(\mathbb{C})} \leq C$ for all $\varepsilon > 0$.*

The constant C in (b) *depends only on the norm of the Cauchy transform bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$, and conversely.*

This lemma is a particular case of a result which applies to more general linear operators. The statement (b) should be understood as a weak substitute of the $L^\infty(\mu)$ boundedness of the Cauchy integral operator, which does not hold in general.

We will prove the easy implication of the lemma, that is, (b) $\Rightarrow$ (a). For the other implication, which is due to Davie and Øksendal [DØ] the reader is referred to [Ch, Chapter VII] and [To1].

Proof of (b)⇒(a). It is enough to show that for any complex measure $\nu \in M(\mathbb{C})$ and any $\lambda > 0$,

$$\mu\{x \in \mathbb{C} : \mathrm{Re}(\mathcal{C}_\varepsilon \nu(x)) > \lambda\} \leq \frac{C\|\nu\|}{\lambda}.$$

To this end, let us denote by A the set on the left side above, and let h be a function supported on A fulfilling the properties in the statement (b) of the lemma. Then we have

$$\begin{aligned}\mu(A) &\leq C\int h\, d\mu \leq \frac{C}{\lambda}\,\mathrm{Re}\Big(\int (\mathcal{C}_\varepsilon \nu) h\, d\mu\Big)\\ &= \frac{-C}{\lambda}\,\mathrm{Re}\Big(\int \mathcal{C}_\varepsilon(h\, d\mu)\, d\nu\Big) \leq \frac{C\|\nu\|}{\lambda}.\end{aligned}$$

□

Remark 18. Notice that if E supports a non zero Radon measure μ with linear growth such that the Cauchy integral operator $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$, then there exists some nonzero function h with $0 \leq h \leq \chi_E$ such that $\|\mathcal{C}_\varepsilon(h\, d\mu)\|_{L^\infty(\mathbb{C})} \leq C$ uniformly on ε, by Theorem 6 and the preceding lemma. Letting $\varepsilon \to 0$, we infer that $|\mathcal{C}(h\, d\mu)(z)| \leq C$ for all $z \notin E$, and so $\gamma(E) > 0$.

A more precise result will be proved in Theorem 19 below.

9. The Painlevé problem and the semiadditivity of analytic capacity

9.1. Semiadditivity of γ_+ and its characterization in terms of curvature. We denote by $\Sigma(E)$ the set of Radon measures supported on E such that $\mu(B(x,r)) \leq r$ for all $x \in \mathbb{C}$, $r > 0$.

Theorem 19. *For any compact set $E \subset \mathbb{C}$ we have*

$$\begin{aligned}\gamma_+(E) &\approx \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ \|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq 1\ \forall\varepsilon > 0\bigr\}\\ &\approx \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ \|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \leq \mu(E)\ \forall\varepsilon > 0\bigr\}\\ &\approx \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ c^2(\mu) \leq \mu(E)\bigr\}\\ &\approx \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\bigr\}.\end{aligned}$$

In the statement above, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)}$ stands for the operator norm of $\mathcal{C}_\mu$ on $L^2(\mu)$. That is, $\|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} = \sup_{\varepsilon>0} \|\mathcal{C}_{\mu,\varepsilon}\|_{L^2(\mu),L^2(\mu)}$.

Proof. We denote

$$\begin{aligned}S_1 &:= \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ \|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq 1\ \forall\varepsilon > 0\bigr\},\\ S_2 &:= \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ \|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \leq \mu(E)\ \forall\varepsilon > 0\bigr\},\\ S_3 &:= \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ c^2(\mu) \leq \mu(E)\bigr\},\\ S_4 &:= \sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\bigr\}.\end{aligned}$$

We will show that $\gamma_+(E) \lesssim S_1 \lesssim S_2 \approx S_3 \lesssim S_4 \lesssim \gamma_+(E)$. We will give two proofs of $S_3 \lesssim S_4$. One uses the $T(1)$ theorem and the other not (and so it is more elementary).

Proof of $\boldsymbol{\gamma_+(E) \lesssim S_1}$. Let μ be supported on E such that $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1$ with $\gamma_+(E) \leq 2\mu(E)$. It is enough to show that μ has linear growth and $\|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq C$ uniformly on $\varepsilon > 0$.

First we will prove the linear growth of μ. For any fixed $x \in \mathbb{C}$, by Fubini it turns out that for almost all $r > 0$,

$$\int_{|z-x|=r} \frac{1}{|z-x|}\, d\mu(z) < \infty.$$

For this r we have

$$\mu(B(x,r)) = -\int_{|z-x|=r} \mathcal{C}\mu(z)\frac{dz}{2\pi\mathrm{i}} \leq r.$$

Now the linear growth of μ follows easily.

To deal with the $L^\infty(\mu)$ norm of $\mathcal{C}_\varepsilon$ we use a standard technique: we replace $\mathcal{C}_\varepsilon$ by the regularized operator $\widetilde{\mathcal{C}}_\varepsilon$, defined as

$$\widetilde{\mathcal{C}}_\varepsilon\mu(x) = \int r_\varepsilon(y-x)\,d\mu(y),$$

where r_ε is the kernel

$$r_\varepsilon(z) = \begin{cases} \dfrac{1}{z} & \text{if } |z| > \varepsilon, \\ \dfrac{\overline{z}}{\varepsilon^2} & \text{if } |z| \leq \varepsilon. \end{cases}$$

Then, $\widetilde{\mathcal{C}}_\varepsilon\mu$ is the convolution of the complex measure μ with the uniformly continuous kernel r_ε and so $\widetilde{\mathcal{C}}_\varepsilon\mu$ is a continuous function. Also, we have

$$r_\varepsilon(z) = \frac{1}{z} * \frac{\chi_\varepsilon}{\pi\varepsilon^2},$$

where χ_ε is the characteristic function of $B(0,\varepsilon)$. Since μ is compactly supported, we have the following identity:

$$\widetilde{\mathcal{C}}_\varepsilon\mu = \frac{1}{z} * \chi_\varepsilon\pi\varepsilon^2 * \mu = \frac{\chi_\varepsilon}{\pi ve^2} * \mathcal{C}\mu.$$

This equality must be understood in the sense of distributions, with $\mathbb{C}\mu$ being a function of $L^1_{\text{loc}}(\mathcal{C})$ with respect to Lebesgue planar measure. As a consequence, if $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1$, we infer that $\|\widetilde{\mathcal{C}}_\varepsilon\mu\|_{L^\infty(\mu)} \leq 1$ for all $\varepsilon > 0$.

Since μ has linear growth, we have

$$|\widetilde{\mathcal{C}}_\varepsilon\mu(x) - \mathcal{C}_\varepsilon\mu(x)| = \frac{1}{\varepsilon^2}\Big|\int_{|y-x|<\varepsilon} (\overline{y-x})\,d\mu(y)\Big| \leq C,$$

and so $\|\mathcal{C}_\varepsilon\mu\|_{L^\infty(\mu)} \leq C$ uniformly on $\varepsilon > 0$.

Proof of $S_1 \lesssim S_2$. Trivial.

Proof of $S_2 \approx S_3$. This is a direct consequence of Proposition 12.

Proof of $S_3 \lesssim S_4$ using the $T(1)$ theorem. Let μ supported on E with linear growth such that $c^2(\mu) \leq \mu(E)$ and $S_3 \leq 2\mu(E)$. We set

$$A := \big\{x \in E : c^2_\mu(x) \leq 2\big\}.$$

By Chebyshev $\mu(A) \geq \mu(E)/2$. Moreover, for any set $B \subset \mathbb{C}$,

$$c^2(\mu_{|B\cap A}) \leq \iiint_{x\in B\cap A} c^2(x,y,z)\,d\mu(x)d\mu(y)d\mu(z)$$
$$= \int_{x\in B\cap A} c_\mu^2(x)\,d\mu(x) \leq 2\mu(B).$$

In particular, this estimate holds when B is any square in $\mathbb{C}$, and so $\mathcal{C}_{\mu_{|A}}$ is bounded on $L^2(\mu_{|A})$, by Theorem 14. Thus $S_4 \gtrsim \mu(A) \approx S_3$.

Proof of $S_3 \lesssim S_4$ using Lemma 15. Take μ supported on E with linear growth such that $c^2(\mu) \leq \mu(E)$ and $S_3 \leq 2\mu(E)$. By Proposition 12, we deduce that $\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} \leq C\mu(E)$ uniformly on $\varepsilon > 0$. By Lemma 15, there exists $G \subset E$ with $\mu(G) \geq \mu(E)/4$ and such that $\mathcal{C}_{\mu_{|G}} : L^2(\mu_{|G}) \to L^2(\mu_{|G})$ is bounded with norm bounded above by some absolute constant. Thus, the measure $\nu = \mu_{|G}$ is supported on E, has linear growth, and satisfies $\nu(E) \geq \mu(E)/4$ and $\|\mathcal{C}_\nu\|_{L^2(\nu),L^2(\nu)} \leq\ C$.

Proof of $S_4 \lesssim \gamma_+(E)$. This is a direct consequence of Lemma 17 and the fact that the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ implies its boundedness from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$, as shown in Theorem 6. □

From the preceding theorem, since the term

$$\sup\bigl\{\mu(E) : \mu \in \Sigma(E),\ \|\mathcal{C}_\mu\|_{L^2(\mu),L^2(\mu)} \leq 1\bigr\}$$

is countably semiadditive, we deduce that γ_+ is also countably semiadditive.

Corollary 20. *The capacity γ_+ is countably semiadditive. That is, if E_i, $i = 1, 2, \ldots$, is a countable (or finite) family of compact sets, we have*

$$\gamma_+\left(\bigcup_{i=1}^\infty E_i\right) \leq C\sum_{i=1}^\infty \gamma_+(E_i).$$

The semiadditivity of γ_+ and its characterization in terms of curvature were proven in [To1], via the $T(1)$ theorem for the Cauchy transform.

Another consequence of Theorem 19 is that the capacity γ_+ can be characterized in terms of the following potential, introduced by Verdera [Ve2]:

$$U_\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{r} + c_\mu^2(x)^{1/2}. \tag{42}$$

The precise result is the following.

Corollary 21. *For any compact set $E \subset \mathbb{C}$ we have*

$$\gamma_+(E) \approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\ U_\mu(x) \leq 1\ \forall x \in \mathbb{C}\big\}.$$

The proof of this corollary follows easily from the fact that

$$\gamma_+(E) \approx \sup\big\{\mu(E) : \mu \in \Sigma(E),\ c^2(\mu) \leq \mu(E)\big\},$$

using Chebyshev. The details are left for the reader.

Let us remark that the preceding characterization of γ_+ in terms of U_μ is interesting because it suggests that some techniques of potential theory could be useful to study γ_+. See [To7] and [Ve2].

9.2. Comparability between $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma_+}$. In [To8] the following result has been proved.

Theorem 22. *There exists an absolute constant C such that for any compact set $E \subset \mathbb{C}$ we have*

$$\gamma(E) \leq C\gamma_+(E).$$

As a consequence, $\gamma(E) \approx \gamma_+(E)$.

An obvious corollary of the preceding result and the characterization of γ_+ in terms of curvature obtained in Theorem 19 is the following.

Corollary 23. *Let $E \subset \mathbb{C}$ be compact. Then, $\gamma(E) > 0$ if and only if E supports a non zero Radon measure with linear growth and finite curvature.*

Since we know that γ_+ is countably semiadditive, the same happens with γ:

Corollary 24. *Analytic capacity is countably semiadditive. That is, if E_i, $i = 1, 2, \ldots$, is a countable (or finite) family of compact sets, we have*

$$\gamma\left(\bigcup_{i=1}^{\infty} E_i\right) \leq C \sum_{i=1}^{\infty} \gamma(E_i).$$

Notice that, by Theorem 19, to prove Theorem 22 it is enough to show that there exists some measure μ supported on E with linear growth, satisfying $\mu(E) \approx \gamma(E)$, and such that the Cauchy transform $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$ with absolute constants. To implement this argument, the main tool used in [To8] is the $T(b)$ theorem of Nazarov, Treil and Volberg [NTV3]. To apply this theorem, one has to construct a suitable measure μ and a function $b \in L^\infty(\mu)$ fulfilling some suitable para-accretivity conditions. The construction of μ and b is the main difficulty which is overcome in [To8], by means of a bootstrapping argument which involves the potential U_μ of (42) and an induction on scales.

Let us remark that the comparability between γ and γ_+ had been previously proved by P. Jones for compact connected sets by geometric arguments, very different from the ones in [To8] (see [Pa, Chapter 3]). On the other hand, the case of Cantor sets was studied in [MTV1]. The proof of [To8] is inspired in part by the ideas in [MTV1].

Corollary 23 yields a characterization of removable sets for bounded analytic functions in terms of curvature of measures. Although this result has a definite geometric flavour, it is not clear if this is a really good geometric characterization. Nevertheless, in [To11] it has been shown that the characterization is invariant under bilipschitz mappings, using a corona type decomposition for non doubling measures. See also [GV] for an analogous result for some Cantor sets.

9.3. Analytic capacity, the Cauchy transform, and rectifiability. In this subsection we will give a brief summary of the relationship between the Cauchy transform, analytic capacity and rectifiability.

A set is called rectifiable if it is $\mathcal{H}^1$-almost all contained in a countable union of rectifiable curves. Recall that $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure. On the other hand, it is purely unrectifiable if it intersects any rectifiable curve at most in a set of zero length.

David and Léger [Lé] proved the following nice result:

Theorem 25. *Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. If $c^2(\mathcal{H}^1_{|E}) < \infty$, then E is rectifiable.*

The proof of Theorem 25 in [Lé] uses geometric techniques, in the spirit of the ones used by P. Jones in [Jo] and by David and Semmes in [DS].

As a corollary of the preceding theorem, using Proposition 12, one infers that if the Cauchy transform is bounded in $L^2(\mathcal{H}^1_{|E})$, then E is rectifiable. A more quantitative version of this result proved by Mattila, Melnikov and Verdera [MMV] asserts that if E is AD-regular (i.e., $\mathcal{H}^1(E \cap B(x,r)) \approx r$ for $x \in E$ and $0 < r \leq \operatorname{diam}(E)$) and the Cauchy transform is bounded on $L^2(\mathcal{H}^1_{|E})$, then E is contained in an AD regular curve Γ.

Recall that David's Theorem 16 (the solution of Vitushkin's conjecture) asserts if $E \subset \mathbb{C}$ has finite length then, $\gamma(E) = 0$ if and only if E is purely unrectifiable. To prove Vitushkin's conjecture, David proved a suitable $T(b)$ type theorem valid for non doubling measures (using a preliminary result from [DM]). Using this theorem, he was able to show that if E has finite length and positive analytic capacity, then it contains a subset F with positive length such that the Cauchy transform is bounded in $L^2(\mathcal{H}_{|F})$. As noticed above, this implies that F is rectifiable.

9.4. Other capacities. In [To9], some results analogous to Theorems 19 and 22 have been obtained for the continuous analytic capacity α. This capacity, introduced by VITUSHKIN (see [Vi2]), is defined like γ in (39), with the additional requirement that the functions f considered in the sup should extend continuously to the whole complex plane. In particular, in [To9] it is shown that α is semiadditive. This result has some nice consequences for the theory of uniform rational approximation on the complex plane. For example, it implies the so called *inner boundary conjecture*.

VOLBERG [Vo] has proved the natural generalization of Theorem 22 to higher dimensions. In this case, one should consider the Lipschitz harmonic capacity instead of the analytic capacity (see [MP] for the definition and properties of Lipschitz harmonic capacity). The main difficulty arises from the fact that in this case one does not have any good substitute of the notion of curvature of measures, and then one has to argue with a potential very different from the one defined in (42). By combining the arguments in [To9] and [Vo] one can prove the semiadditivity of the so called $\mathcal{C}^1$ harmonic capacity, introduced by PARAMONOV [Par] because its application to problems of $\mathcal{C}^1$ harmonic approximation (see [RT]). See also [MT] and [To14] for related results which avoid the use of any notion similar to curvature.

The techniques in Theorem 22 have also been used by PRAT [Pr1], [Pr2], and MATEU, PRAT and VERDERA [MPV] to study the capacities γ_α associated to α-dimensional signed Riesz kernels with α non integer:

$$k(x,y) = \frac{y-x}{|y-x|^{\alpha+1}}.$$

In [Pr2], it is shown that these capacities are semiadditive and comparable to their positive versions $\gamma_{\alpha,+}$, analogously to analytic capacity. However, there are some big differences between the behavior of analytic capacity and the capacities γ_α, $\alpha \notin \mathbb{Z}$. For instance, in [Pr1] it is shown that sets with finite α-dimensional Hausdorff measure have vanishing capacity γ_α when $0 < \alpha < 1$. Moreover, for these α's it is proved in [MPV] that γ_α is comparable to one of the non linear Wolff's capacities. The case of non integer α with $\alpha > 1$ seems much more difficult to study, although in the AD-regular situation some results have been obtained [Pr1]. The results in [Pr1] and [MPV] show that the behavior of γ_α with α non integer is very different from the one with α integer.

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