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SINGULAR INTEGRALS AND STRONG MAXIMAL FUNCTIONS IN WEIGHTED GRAND LEBESGUE SPACES

Vakhtang Kokilashvili

Abstract. In this lecture we will discuss the weighted boundedness problem for various integral operators in generalized grand Lebesgue spaces $L^{p),\theta}_w$. Namely, a complete description of weight functions governing one-weight inequalities for multiple singular integrals and strong maximal functions will be presented.

1. The grand Lebesgue spaces and their generalizations

The grand Lebesgue spaces were introduced by T. Iwaniec and C. Sbordone [5]. Nowadays, the theory of these spaces and associated small Lebesgue spaces is one of the intensively developing directions of modern analysis. The necessity to investigate these spaces emerged from their rather essential role in various fields, in particular, in the integrability problem of Jacobian under minimal hypothesis (T. Iwaniec and C. Sbordone, P. Koskela and X. Zhong), in the study of maximal operators and, more generally, quasi-linear operators (A. Fiorenza and M. Krbec), in extrapolation theory (M. Milman), in variation problems (T. Iwaniec and C. Sbordone), in regularity and uniqueness problems in grand Sobolev spaces for parabolic equations with measure data (A. Fiorenza, A. Mercaldo and J. M. Rakotoson). In the theory in PDE, it turns out that they are the right spaces in which some nonlinear equations have been considered (A. Fiorenza and C. Sbordone, L. Greco etc.).

Let G be a bounded Lebesgue measurable set in $\mathbb{R}^n$. The grand Lebesgue space $L^{p)}(G)$ $(1 < p < \infty)$ is a rearrangement invariant non-reflexive Banach

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space defined by the norm

$$\|f\|_{L^{p)}(G)} := \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|G|} \int_G |f(t)|^{p-\varepsilon}\, dt \right)^{\frac{1}{p-\varepsilon}}.$$

It is worth mentioning that the following continuous embeddings hold:

$$L^p(G) \hookrightarrow L^{p)}(G) \hookrightarrow L^{p-\varepsilon}(G), \quad 0<\varepsilon\le p-1. \tag{1.1}$$

Let w be a weight, i.e., Lebesgue integrable a.e. positive function on G. We denote by $L^q_w(G)$ $(1<q<\infty)$ the space of all measurable functions on G for which

$$\|f\|_{L^q_w(G)} := \left(\frac{1}{|G|} \int_G |f(t)|^q w(t)\, dt \right)^{\frac{1}{q}} < \infty.$$

It is known that the space $L^{p)}_w(G)$ defined by the norm

$$\|f\|_{L^{p)}_w(G)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w(G)}, \quad 1<p<\infty,$$

is a Banach function space. Except for the trivial case of $w\equiv$ const the space $L^{p)}_w(G)$ is not rearrangement-invariant (see, e.g., [2]).

Let φ be positive increasing function on $(0,p-1)$, $1<p<\infty$. We define

$$L^{p),\varphi}_w(G) := \left\{ f : \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}_w(G)} < \infty \right\}.$$

In the case $\varphi(\varepsilon)=\varepsilon^\theta$, $\theta>0$ and $w\equiv$ const we have the generalized grand Lebesgue space denoted by $L^{p),\theta}(G)$. For these spaces and some applications we refer to [3]. When $w\neq$ const we have the weighted space and denote it by $L^{p),\theta}_w(G)$. It is known that $L^{p),\theta}_w(G)$ is a Banach function space and that $L^p_w(G)\subset L^{p),\theta_1}_w(G)\subset L^{p),\theta_2}_w(G)$ whenever $\theta_1<\theta_2$ (see, e.g., [3]). It is clear that for $\theta=0$,

$$\|f\|_{L^{p),\theta}_w(G)} \equiv \|f\|_{L^p_w(G)},$$

the norm in the weighted Lebesgue space.

2. The multiple Calderón singular integral and strong maximal function

The impulse for the study of mapping properties of classical integral operators in grand Lebesgue spaces comes from A. Fiorenza, B. Gupta and P. Jain [2] who had proved the boundedness of the Hardy-Littlewood maximal operator in weighted spaces.

Definition 2.1. Let $1 < p < \infty$. Let $\mathbb{J}$ be an n-dimensional parallelepiped with sides parallel to coordinate axes. We say that a weight w belongs to the class $\mathcal{A}_p(\mathbb{J})$ ($w \in \mathcal{A}_p(\mathbb{J})$) if

$$\mathcal{A}_p(w,\mathbb{J}) := \sup_{J\subset\mathbb{J}}\left(\frac{1}{|J|}\int_J w(x)\,dx\right)\left(\frac{1}{|J|}\int_J w^{1-p'}(t)\,dt\right)^{p-1} < \infty, \tag{2.1}$$

where the supremum is taken over all n-dimensional parallelepipeds J with sides parallel to coordinate axis contained in $\mathbb{J}$.

B. Muckenhoupt [10] proved that the Hardy-Littlewood maximal operator is bounded in weighted classical Lebesgue spaces L^p_w if and only if $w \in A_p$, i.e., there is a positive constant C such that for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}(t)\,dt\right)^{p-1} \le C < \infty.$$

Later R. Hunt, B. Muckenhoupt and R. L. Wheeden [4] established that necessary and sufficient condition for the boundedness of the Hilbert transform in L^p_w is that w belongs to the class A_p defined on the real line.

Let $\mathbb{J} := I_1 \times I_2 \times \cdots \times I_n$, where I_j, $j = 1,\dots,n$, are closed intervals in $\mathbb{R}$. Our purpose in this section is to characterize the boundedness of multiple Calderón singular operator

$$\mathcal{C}^a f(x) = \int_{\mathbb{J}} \prod_{i=1}^{n} \frac{a_i(x_i) - a_i(t_i)}{(x_i - t_i)^2} f(t)\,dt, \quad x = (x_1,\dots,x_n) \in \mathbb{J},$$

where $a_i \in \mathrm{Lip}\,1$ on I_i, $i = 1,2,\dots,n$. In the case when $n=1$ and $a_j(x_j) \equiv x_j$ this statement was proved in [8]. One-dimensional analogy of Theorem 2.1 was proved in [7]. In the classical weighted Lebesgue spaces similar result was established in [6].

Now we formulate the main results of this section. We begin with general-type theorem:

Theorem 2.1. *Let a linear operator T be bounded in every weighted Lebesgue space $L^p_w(\mathbb{J})$ with $1<p<\infty$ and $w\in\mathcal{A}_p(\mathbb{J})$. Then T is also bounded in the weighted grand Lebesgue space $L^{p),\theta}_w(\mathbb{J})$ for every $1<p<\infty$, $w\in\mathcal{A}_p(\mathbb{J})$ and $\theta>0$.*

The next statement is also valid:

Theorem 2.2. *Let $1<p<\infty$ and let $\theta>0$. Then $\mathcal{C}^a$ is bounded in $L^{p),\theta}_w(\mathbb{J})$ if $w\in\mathcal{A}_p(\mathbb{J})$. Conversely, if there exists a positive constant m such that $0<m\le|a'_j(x_j)|$, $j=1,2,\dots,n$, for a.a. $x\in\mathbb{J}$, then the condition $w\in\mathcal{A}_p(\mathbb{J})$ is also necessary for the boundedness of $\mathcal{C}^a$ in $L^{p),\theta}_w(\mathbb{J})$.*

Let us now discuss the following strong maximal operator

$$\mathcal{M}_r f(x)=\sup_{J\ni x}\left(\frac{1}{|J|}\int_J |f(y)|^r\,dy\right)^{\frac{1}{r}},\quad r\ge 1,\ x\in\mathbb{J},$$

where the supremum is taken over all n-dimensional parallelepipeds in $\mathbb{J}$ with sides parallel to the coordinate axes. The next theorem holds:

Theorem 2.3. *Let $1<p<\infty$ and let $\theta>0$. Then the strong maximal operator $\mathcal{M}_r$ is bounded in $L^{p),\theta}_w$ if and only if $w\in\mathcal{A}_p(\mathbb{J})$.*

Now we give sketches of proofs of the main statements of this section. The next lemmas will be useful for us.

Lemma 2.1. *Let $1<p<\infty$ and let w be a weight on $\mathbb{J}$. Suppose that T is a linear operator acting boundedly simultaneously in $L^p_w(\mathbb{J})$ and $L^{p-\sigma}_w(\mathbb{J})$ with some positive number σ, $\sigma<p-1$. Then T is bounded in $L^{p)}_w(\mathbb{J})$.*

Lemma 2.2. *Let $1<p<\infty$ and let w be a weight on $\mathbb{J}$. Then there is a positive constant c such that for all $f\in L^p_w(\mathbb{J})$ and all parallelepipeds $P\subset\mathbb{J}$, the inequality*

$$\|f\chi_P\|_{L^{p)}_w(\mathbb{J})}\le c w(P)^{-1/p}\|f\chi_P\|_{L^p_w(\mathbb{J})}\|\chi_P\|_{L^{p)}_w(\mathbb{J})}$$

holds.

Proof of Lemma 2.1. By the hypothesis of the lemma, we have that there are positive constants c_1 and c_2 independent of f such that

$$\|Tf\|_{L^p_w(\mathbb{J})}\le c_1\|f\|_{L^p_w(\mathbb{J})}$$

and

$$\|Tf\|_{L^{p-\sigma}_w(\mathbb{J})}\le c_2\|f\|_{L^{p-\sigma}_w(\mathbb{J})}.$$

By the Riesz-Thorin theorem, we have that

$$\|Tf\|_{L_w^{p-\varepsilon}(\mathbb{J})} \le c\|f\|_{L_w^{p-\varepsilon}(\mathbb{J})}, \quad 0 \le \varepsilon \le \sigma, \tag{2.2}$$

where the positive constant c does not depend on f and ε.

Let us fix $\varepsilon \in (\sigma, p-1)$. Then, using Hölder's inequality with respect to the exponent $\frac{p-\sigma}{p-\varepsilon}$ and observing that $\left(\frac{p-\sigma}{p-\varepsilon}\right)' = \frac{p-\sigma}{\varepsilon-\sigma}$, we find that

$$\|Tf\|_{L_w^{p-\varepsilon}(\mathbb{J})} \le c_{\mathbb{J}}\left(\int_{\mathbb{J}} |Tf(x)|^{p-\sigma} w(x)\,dx\right)^{\frac{1}{p-\sigma}} w(\mathbb{J})^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}}, \tag{2.3}$$

where

$$c_{\mathbb{J}} := \max\bigl\{1, |\mathbb{J}|^{-1}\bigr\}.$$

Further, the conditions $\sigma < p-1$ and $\varepsilon \in (\sigma, p-1)$ yield that

$$0 < \frac{\varepsilon - \sigma}{(p-\sigma)(p-\varepsilon)} < \frac{p-1-\sigma}{p-\sigma}. \tag{2.4}$$

Let us denote

$$w(\mathbb{J}) := \int_{\mathbb{J}} w.$$

Applying (2.2)–(2.4), we find that

$$\begin{aligned}
\|Tf\|_{L_w^{p)}(\mathbb{J})} &= \max\Bigl\{\sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L_w^{p-\varepsilon}(\mathbb{J})}, \sup_{\sigma<\varepsilon\le p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L_w^{p-\varepsilon}(\mathbb{J})}\Bigr\} \\
&\le \max\Bigl\{\sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L_w^{p-\varepsilon}(\mathbb{J})}, c_{\mathbb{J}} \sup_{\sigma<\varepsilon\le p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L_w^{p-\sigma}} w(\mathbb{J})^{\frac{p-1-\sigma}{p-\sigma}}\Bigr\} \\
&\le c\max\Bigl\{1, \sup_{\sigma<\varepsilon\le p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \sigma^{-\frac{\theta}{p-\sigma}} w(\mathbb{J})^{\frac{p-1-\sigma}{p-\sigma}}\Bigr\} \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|Tf\|_{L_w^{p-\varepsilon}(\mathbb{J})} \\
&\le c\max\Bigl\{1, (p-1)^{\theta} \sigma^{-\frac{\theta}{p-\sigma}} (1+w(\mathbb{J}))^{\frac{p-1-\sigma}{p-\sigma}}\Bigr\} \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}(\mathbb{J})} \\
&\le c(p-1)^{\theta} \sigma^{-\frac{\theta}{p-\sigma}} (1+w(\mathbb{J}))^{\frac{p-1-\sigma}{p-\sigma}} \|f\|_{L_w^{p)}(\mathbb{J})}.
\end{aligned}$$

□

Lemma 2.2 in the one-dimensional case was proved in [1]. The proof for n-dimensional parallelepipeds is similar.

Proof of Theorem 2.1. *Sufficiency* is a consequence of Lemma 2.1 because the class $\mathcal{A}_p$ has the property: $w \in \mathcal{A}_p(\mathbb{J}) \Longrightarrow w \in \mathcal{A}_{p-\sigma}(\mathbb{J})$ for some small positive number σ (see, e.g., [1]).

Necessity. Using Lemma 2.2 and choosing appropriate test functions, we can derive that (2.1) holds for all n-dimensional parallelepipeds J in $\mathbb{J}$ having sufficiently small diameters. Now the result follows immediately. □

Proof of Theorem 2.3 is similar to that of Theorem 2.2; therefore it is omitted.

3. Weighted multiple singular operators and strong maximal functions

For given linear operator T, we consider the weighted operator

$$T_w\colon f \to wT\Big(\frac{f}{w}\Big).$$

Boundedness of the operator T_w in $L^p(\mathbb{J})$, $1<p<\infty$, is equivalent to the boundedness of T in the function space defined by the norm

$$\|\varphi w\|_{L^{p,\theta}(\mathbb{J})}.$$

Note that for grand Lebesgue space the equivalence

$$f \in L^{p)}_w(\mathbb{J}) \Longleftrightarrow f w^{1/p} \in L^{p)}(\mathbb{J})$$

does not hold.

The following theorem is valid:

Theorem 3.1. *Let a weighted operator T_w be bounded in every Lebesgue space $L^p(\mathbb{J})$ with $1<p<\infty$ and a weight w satisfying the condition $w^p \in \mathcal{A}_p(\mathbb{J})$. Then T_w is also bounded in $L^{p),\theta}(\mathbb{J})$ for every $1<p<\infty$, $w^p\in\mathcal{A}_p(\mathbb{J})$ and $\theta>0$.*

Proof. We start with the observation that there exist numbers $\sigma\in(0,p-1)$ and $M>0$ such that

$$\|T_w\|_{L^{p-\varepsilon}(\mathbb{J})\to L^{p-\varepsilon}(\mathbb{J})} \le M \tag{3.1}$$

for arbitrary ε, $0<\varepsilon<\sigma$. Indeed, it is known that if $w^p\in\mathcal{A}_p(\mathbb{J})$ then there exists such $\sigma\in(0,p-1)$ that $w^p\in\mathcal{A}_{p-\sigma}(\mathbb{J})$ and also $w^{p\alpha}\in\mathcal{A}_{p-\sigma}(\mathbb{J})$ for arbitrary α, $0<\alpha<1$ (see, for example, [6]). Let now $\alpha=\frac{p-\sigma}{p}$. Then $0<\alpha<1$ and $w^{p-\sigma}\in\mathcal{A}_{p-\sigma}(\mathbb{J})$.

According to the assumption of the theorem, we have

$$\|T_w f\|_{L^p(\mathbb{J})} \le M_1\|f\|_{L^p(\mathbb{J})}.$$

Also,

$$\|T_w f\|_{L^{p-\sigma}(\mathbb{J})} \le M_2\|f\|_{L^{p-\sigma}(\mathbb{J})}.$$

For $\varepsilon \in (0, \sigma)$, there exists $t_\varepsilon \in (0, 1)$ such that

$$\frac{1}{p-\varepsilon} = \frac{t_\varepsilon}{p} + \frac{1-t_\varepsilon}{p-\sigma}.$$

By the Riesz-Thorin interpolation theorem, the inequality

$$\|T_w f\|_{L^{p-\varepsilon}(\mathbb{J})} \leq M_1^{t_\varepsilon} M_2^{1-t_\varepsilon} \|f\|_{L^{p-\varepsilon}(\mathbb{J})}$$

is fulfilled for arbitrary $\varepsilon \in (0, \sigma)$.

From the latter inequality it follows (3.1).

Further, it is clear that

$$\|T_w f\|_{L^{p),\theta}(\mathbb{J})} = \max\{A, B\}, \tag{3.2}$$

where

$$A = \sup_{0<\varepsilon\leq\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|T_w f\|_{L^{p-\varepsilon}(\mathbb{J})}$$

and

$$B = \sup_{\sigma<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|T_w f\|_{L^{p-\varepsilon}(\mathbb{J})}.$$

Let us fix $\varepsilon \in (\sigma, p-1)$. Then

$$\frac{p-\sigma}{p-\varepsilon} > 1$$

and also

$$0 < \frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)} < \frac{p-1-\sigma}{p-\sigma}.$$

Consequently, using Hölder's inequality with respect to the exponent $(p-\sigma)/(p-\varepsilon)$ and observing that

$$\left(\frac{p-\sigma}{p-\varepsilon}\right)' = \frac{p-\sigma}{\varepsilon-\sigma},$$

we find that

$$\|T_w f\|_{L^{p-\varepsilon}} \leq \|T_w f\|_{L^{p-\sigma}(\mathbb{J})} |\mathbb{J}|^{\frac{\varepsilon-\sigma}{(p-\varepsilon)(p-\sigma)}} \leq c\|T_w f\|_{L^{p-\sigma}(\mathbb{J})},$$

where c is a constant independent of ε and f.

Thus,

$$\begin{aligned} B &\leq c \sup_{\sigma<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \sigma^{-\frac{\theta}{p-\sigma}} \sigma^{\frac{\theta}{p-\sigma}} \|T_w f\|_{L^{p-\sigma}(\mathbb{J})} \\ &\leq cA \max\bigl\{1, (p-1)^\theta \sigma^{-\frac{\theta}{p-\sigma}}\bigr\}. \end{aligned}$$

Now, taking advantage of (3.1), we conclude

$$\|T_w f\|_{L^{p),\theta}(\mathbb{J})} \leq c \max\bigl\{1, (p-1)^\theta \sigma^{-\frac{\theta}{p-\sigma}}\bigr\} \|f\|_{L^{p),\theta}(\mathbb{J})}.$$

□

Corollary 2.1. *The weighted operator*

$$\mathcal{C}_w^a\colon f \mapsto w\mathcal{C}^a\Big(\frac{f}{w}\Big)$$

is bounded in $L^{p),\theta}(\mathbb{J})$ *for arbitrary* p, $1<p<\infty$, $\theta>0$ *and* $w^p \in \mathcal{A}_p(\mathbb{J})$.

The similar proposition is true for weighted strong maximal function:

$$\mathcal{M}_{r,w} f := w\mathcal{M}_r(f/w).$$

Remark 3.1. Theorem 3.1 remains valid if for given p, $1<p<\infty$, we suppose that T_w is bounded in $L^p(\mathbb{J})$ with $w^p \in \mathcal{A}_p(\mathbb{J})$, and also it is bounded in $L^{p-\varepsilon}(\mathbb{J})$ with $w^{p-\varepsilon} \in \mathcal{A}_{p-\varepsilon}(\mathbb{J})$ for some small $\varepsilon > 0$.

Finally, we notice that the boundedness criteria of power weighted Cauchy singular integral operator in grand Lebesgue spaces is established in [9].

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