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QUANTITATIVE PROPERTIES OF QUADRATIC SPLINE WAVELET BASES IN HIGHER DIMENSIONS

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Abstract

To use wavelets efficiently to solve numerically partial differential equations in higher dimensions, it is necessary to have at one's disposal suitable wavelet bases. Ideal wavelets should have short supports and vanishing moments, be smooth and known in closed form, and a corresponding wavelet basis should be well-conditioned. In our contribution, we compare condition numbers of different quadratic spline wavelet bases in dimensions $d = 1, 2$ and 3 on tensor product domains $(0, 1)^d$.

1. Introduction

In recent years, several promising constructions of wavelets were proposed. We mention, for example, a construction of spline-wavelet bases on the interval proposed in [1]. Their bases are compactly supported and generate multiresolution analyses on the unit interval with the desired numbers of vanishing wavelet moments for primal and dual wavelets. Moreover, dual wavelets are also compactly supported. Here, we use recently proposed wavelets based on quadratic splines [2, 3, 4, 5] and propose one other modification. These wavelets have shorter supports, are better conditioned but dual wavelets are not compactly supported. Due to their properties these wavelet bases can be used in the wavelet Galerkin method as well as in adaptive wavelet methods for solving second-order elliptic problems with homogeneous Dirichlet boundary conditions.

2. Wavelet bases

We consider here families $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} \subset L_2(0, 1)$ of functions (wavelets). Let $\mathcal{J}$ be an infinite index set and $\mathcal{J} = \mathcal{J}_\Phi \cup \mathcal{J}_\Psi$, where $\mathcal{J}_\Phi$ is a finite set representing scaling functions living on the coarsest scale. Any index $\lambda \in \mathcal{J}$ is of the form $\lambda = (j, k)$, where $|\lambda| = j$ denotes a scale and k denotes spatial location. At last, for $s \geq 0$ the space H^s will denote a closed subspace of the Sobolev space $H^s(0,1)$, defined e.g. by imposing homogeneous boundary conditions at one or both endpoints, and for $s < 0$ the space H^s will denote the dual space $H^s := (H^{-s})'$. $\|.\|_{H^s}$ will denote the corresponding norm. Further $l_2(\mathcal{J})$ will denote the space consisting of the power summable sequences and $\|.\|_{l_2(\mathcal{J})}$ will denote the corresponding norm.

A family $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} \subset L_2(0,1)$ is called a *wavelet basis* of H^s for some $\gamma, \widetilde{\gamma} > 0$ and $s \in (-\widetilde{\gamma}, \gamma)$, if

- Ψ is a Riesz basis of H^s, that means Ψ forms a basis of H^s and there exist constants $c_s, C_s > 0$ such that for all $\mathbf{b} = \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l_2(\mathcal{J})$ holds

$$c_s \|\mathbf{b}\|_{l_2(\mathcal{J})} \leq \left\|\mathbf{b}^T \Psi\right\|_{H^s} \leq C_s \|\mathbf{b}\|_{l_2(\mathcal{J})},$$

  where $\sup c_s$, $\inf C_s$ are called Riesz bounds and $\mathrm{cond}\Psi := \dfrac{\inf C_s}{\sup c_s}$ is called the condition number of Ψ.

- Functions are local in the sense that $\mathrm{diam}\,(\mathrm{supp}\,\psi_\lambda) \leq C 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where C is a constant independent of λ.

- Functions ψ_λ, $\lambda \in \mathcal{J}_\Psi$, have cancellation properties of order m, i.e.

$$\left|\int_0^1 v(x)\,\psi_\lambda(x)\,dx\right| \leq 2^{-m|\lambda|}\,|v|_{H^m(0,1)}, \qquad \forall v \in H^m(0,1).$$

  It means that integration against wavelets eliminates smooth parts of functions and it is equivalent with vanishing wavelet moments of order m.

3. Scaling functions

As inner scaling functions, we use quadratic B-splines, because they have short support and can be easily adapted to a bounded interval by employing multiple knots at the endpoints. In the case of quadratic basis, there is necessary to add only one boundary scaling function at each boundary to preserve polynomial exactness and homogeneous boundary conditions. Specifically, the quadratic spline function $\phi(x)$ is given by

$$\phi(x) = \begin{cases} \frac{x^2}{2} & x \in [0,1], \\ -x^2 + 3x - \frac{3}{2} & x \in [1,2], \\ \frac{x^2}{2} - 3x + \frac{9}{2} & x \in [2,3], \\ 0 & \text{otherwise} \end{cases}$$

and the left boundary function $\phi_b(x)$ is given by

$$\phi_b(x) = \begin{cases} -\frac{3x^2}{2} + 2x & x \in [0,1], \\ \frac{x^2}{2} - 2x + 2 & x \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

Then a scaling basis satisfying homogeneous Dirichlet boundary conditions is determined by

$$\Phi_j = \left\{ \phi_{j,k} / \left| \phi_{j,k} \right|_{H_0^1(0,1)}, k = 1, \ldots, 2^j \right\},$$

where for $j \geq 2$ and $x \in [0,1]$ we define

$$\begin{aligned} \phi_{j,k}(x) &= 2^{j/2} \phi(2^j x - k + 2), k = 2, ..., 2^j - 1, \\ \phi_{j,1}(x) &= 2^{j/2} \phi_b(2^j x), \quad \phi_{j,2^j}(x) = 2^{j/2} \phi_b(2^j (1-x)). \end{aligned}$$

4. Wavelets and wavelet bases

Constructed wavelets should have small support and vanishing moments and the corresponding wavelet basis should be well-conditioned. Unlike biorthogonal wavelets [1, 6], where primal wavelets have significantly larger support than scaling functions and dual wavelets are local, we focus here on primal wavelets which have the same length of support as scaling functions or shorter. Let us denote the space spanned by the set Φ_j by V_j. Then, we define complement spaces W_j by $V_{j+1} = V_j \oplus W_j$ and a wavelet basis is constructed to be the basis of W_j.

First, we look at inner wavelets with minimal support and with the number of vanishing moments that equals to the degree of used B-splines. The quadratic spline-wavelet is then given by

$$\psi(x) = -\frac{1}{4}\phi(2x) + \frac{3}{4}\phi(2x-1) - \frac{3}{4}\phi(2x-2) + \frac{1}{4}\phi(2x-3).$$

In [2], the boundary wavelet was constructed by prescribing the number of vanishing moments, the support in the interval $[0, 5/2]$, homogeneous Dirichlet boundary conditions and finally, it should be from the space spanned by $\{\phi_b(2x),\, \phi(2x-k) : k \in \mathbb{N}_0\}$. This boundary wavelet is determined by

$$\psi_b^1(x) = -\frac{5}{2}\phi_b(2x) + \frac{47}{12}\phi(2x) - \frac{13}{4}\phi(2x-1) + \phi(2x-2).$$

In [3], we constructed a new boundary wavelet by allowing its support to be in the interval $[0,3]$ and prescribing three vanishing moments, homogeneous Dirichlet boundary conditions and finally, it again should be from the space spanned by $\{\phi_b(2x),\, \phi(2x-k) : k \in \mathbb{N}_0\}$. Then, there are infinitely many solutions and we selected one that has zero wavelet coefficient corresponding to the scaling function $\phi(2x-2)$. Consequently, the arising system has exactly one solution up to multiplication by a constant. This boundary wavelet is given by

$$\psi_b^2(x) = -\frac{15}{2}\phi_b(2x) + \frac{43}{4}\phi(2x) - \frac{27}{4}\phi(2x-1) + \phi(2x-3).$$

In [4], we propose the boundary wavelet prescribing the same properties as above. But we use the free parameter identified in [3] to ensure the orthogonality of constructed boundary wavelet with the nearest inner wavelet:

$$\psi_b^3(x) = -\frac{920}{209}\phi_b(2x) + \frac{3697}{627}\phi(2x) - \frac{569}{209}\phi(2x-1) - \frac{259}{209}\phi(2x-2) + \phi(2x-3).$$

Here, we use also the above mentioned free parameter to ensure the orthogonality of the first derivative of the constructed boundary wavelet with the first derivative of the nearest inner wavelet:

$$\psi_b^4(x) = -\frac{40}{13}\phi_b(2x) + \frac{149}{39}\phi(2x) - \phi(2x-1) - \frac{23}{13}\phi(2x-2) + \phi(2x-3).$$

Further, we look at wavelets with one vanishing moment. An inner wavelet ψ with $\operatorname{supp}\psi = [0.5, 2.5]$ is then given by

$$\psi(x) = -\frac{1}{2}\phi(2x-1) + \frac{1}{2}\phi(2x-2).$$

And a boundary wavelet ψ_b with $\operatorname{supp}\psi_b = [0, 1.5]$ and with one vanishing wavelet moment is defined by:

$$\psi_b(x) = \frac{\phi_b(2x)}{2} - \frac{\phi(2x)}{3}.$$

For $j \geq 2$ and $x \in [0,1]$ we define

$$\begin{aligned} \psi_{j,k}(x) &= 2^{j/2}\psi(2^j x - k + 2), k = 2, ..., 2^j - 1, \\ \psi_{j,1}(x) &= 2^{j/2}\psi_b(2^j x), \quad \psi_{j,2^j}(x) = 2^{j/2}\psi_b(2^j(1-x)). \end{aligned}$$

We denote

$$\Psi_j = \left\{\psi_{j,k} / \left|\psi_{j,k}\right|_{H_0^1(0,1)}, k = 1, \ldots, 2^j\right\}.$$

Then the sets

$$\Psi^s = \Phi_2 \cup \bigcup_{j=2}^{1+s} \Psi_j \quad \text{and} \quad \Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j$$

are a multiscale wavelet basis and a wavelet basis of the space $H_0^1(0,1)$, respectively. The proof can be found in [5]. Multiscale wavelet bases and wavelet bases for other construction can be defined in a similar way.

To define wavelets in higher dimensions we use the tensor product. The tensor product of functions u and v is defined by $(u \otimes v)(x_1, x_2) := u(x_1)\, v(x_2)$. We show an example of such wavelet basis in dimension $d = 2$. We set

$$\begin{aligned} F_j &= \left\{\phi_{j,k} \otimes \phi_{j,l} / \left|\phi_{j,k} \otimes \phi_{j,l}\right|_{H_0^1(\Omega)}, k, l = 1, \ldots, 2^j\right\}, \\ G_j^1 &= \left\{\phi_{j,k} \otimes \psi_{j,l} / \left|\phi_{j,k} \otimes \psi_{j,l}\right|_{H_0^1(\Omega)}, k, l = 1, \ldots, 2^j\right\}, \\ G_j^2 &= \left\{\psi_{j,k} \otimes \phi_{j,l} / \left|\psi_{j,k} \otimes \phi_{j,l}\right|_{H_0^1(\Omega)}, k, l = 1, \ldots, 2^j\right\}, \\ G_j^3 &= \left\{\psi_{j,k} \otimes \psi_{j,l} / \left|\psi_{j,k} \otimes \psi_{j,l}\right|_{H_0^1(\Omega)}, k, l = 1, \ldots, 2^j\right\}, \end{aligned}$$

where $\Omega = (0,1)^2$. Then the sets defined by

$$\Psi_s^{2D} = F_2 \cup \bigcup_{j=2}^{1+s} \left(G_j^1 \cup G_j^2 \cup G_j^3\right), \quad \Psi^{2D} = F_2 \cup \bigcup_{j=2}^{\infty} \left(G_j^1 \cup G_j^2 \cup G_j^3\right) \tag{1}$$

are a wavelet basis and a multiscale wavelet basis of the space $H_0^1(\Omega)$.

5. Condition numbers

We compute condition numbers of stiffness matrices corresponding to the following Dirichlet problem

$$-\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = f \quad \text{in} \;\; \Omega = (0,1)^d \qquad \text{with} \;\; u = 0 \quad \text{on} \;\; \partial\Omega$$

discretized using above mentioned wavelet bases. These condition numbers are closely related to the condition number of wavelet basis of the space $H_0^1(\Omega)$. In all tables, $B3^i$ will denote the wavelet basis with three vanishing wavelet moments and with boundary wavelet ψ_b^i, $B1$ will denote the wavelet basis with one vanishing wavelet moment and finally n will denote a number of used basis functions.

n	$B3^1$	$B3^2$	$B3^3$	$B3^4$	$B1$
8	8.9	5.4	3.9	3.5	2.8
16	10.1	6.1	4.6	4.4	2.8
32	10.6	6.5	4.9	4.7	2.8
64	10.8	6.6	5.0	4.8	2.8
128	10.9	6.7	5.1	4.9	2.8
256	10.9	6.8	5.2	4.9	2.8
512	10.9	6.8	5.2	4.9	2.8
1024	11.0	6.8	5.2	4.9	2.8
2048	11.0	6.9	5.2	4.9	2.8
4096	11.0	6.9	5.2	4.9	2.8

Table 1: Condition numbers for $d = 1$

n	$B3^1$	$B3^2$	$B3^3$	$B3^4$	$B1$
64	56.6	29.3	18.7	21.9	7.5
256	83.6	42.1	29.9	32.6	11.0
1096	98.7	50.2	36.7	38.0	13.6
4096	107.5	55.4	40.6	40.8	15.3
16384	113.1	58.7	43.1	42.5	16.5
65536	116.8	61.0	44.6	43.7	17.3
262144	119.4	62.5	45.7	44.4	17.9
1048576	121.3	63.7	46.5	44.9	18.3

Table 2: Condition numbers for $d = 2$

n	$B3^1$	$B3^2$	$B3^3$	$B3^4$	$B1$
512	470.8	227.7	169.3	208.5	47.4
4096	815,9	402.2	313.2	362.6	85.0
32768	1027,1	500.9	389.1	429.5	113.8
262144	1153,6	552.9	425.0	459.0	132.9
2097152	1230,7	581.8	443.5	474.1	145.3

Table 3: Condition numbers for $d = 3$

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