Cyril Fischer; Ondřej Fischer; Ladislav Frýba

Numerical modelling of a bridge subjected to simultaneous effect of a moving load and a vertical seismic ground excitation

In: Jan Chleboun and Petr Přikryl and Karel Segeth and Jakub Šístek and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 8-13, 2014. Institute of Mathematics AS CR, Prague, 2015. pp. 71--76.

Persistent URL: http://dml.cz/dmlcz/702665

*Programs and Algorithms of Numerical Matematics 17*
J. Chleboun, P. Přikryl, K. Segeth, J. Šístek, T. Vejchodský (Eds.)
Institute of Mathematics AS CR, Prague 2015

# NUMERICAL MODELLING OF A BRIDGE SUBJECTED TO SIMULTANEOUS EFFECT OF A MOVING LOAD AND A VERTICAL SEISMIC GROUND EXCITATION

Cyril Fischer, Ondřej Fischer, Ladislav Frýba

Institute of Theoretical and Applied Mechanics, AS CR, v.v.i.
Prosecká 76, Prague 9, Czech Republic
{fischerc|fischero|fryba}@itam.cas.cz

**Abstract**

A simple beam subjected to a row of regularly distributed moving forces and simultaneous vertical motions of its supports is described using a simplified theoretical model and a finite differences approach. Several levels of simplification of the structure and input data are supposed. Numerical results confirm legitimacy of the assumptions.

## 1. Introduction

Although dynamic action of moving loads on structures was studied since middle of the nineteenth century, the combined effect of train and earthquake attracted attention only recently, [5]. In the present work, we concentrate to the problem of vertical vibrations of a beam, which is subjected to a row of regularly spaced fast moving forces and simultaneously to motion of its supports due to an earthquake.

An approximative analytical solution to the problem was formulated at the cost of significant simplification many times, e.g., [2]. However, these formulae bring their own difficulties for numerical enumeration: they involve partial sums of infinite trigonometric series, which can introduce spurious oscillations, or hidden pairs of terms, which cancel themselves under certain conditions and thus they are a potential source of numerical instability. Moreover, simplifying assumptions like lack of damping or a limited number of eigenmodes taken into account lower credibility of the formulae. Such obstacles divert attention to numerical alternatives.

Numerical algorithms for solution to fourth order parabolic PDEs have a long tradition. The available methods comprise explicit and implicit finite difference schemas or several variants of finite element methods. Method of lines gained in popularity for general problems. It reformulates the PDE to the form, which is convenient for application of a standard ODE solver.

In this paper, we present an attempt to employ an implicit difference schema for solution to the PDE describing the transverse vibrations of a beam. The numerical procedure is tested on the benchmark case introduced by Evans in [1] and on a simple model of a real bridge, see [3].

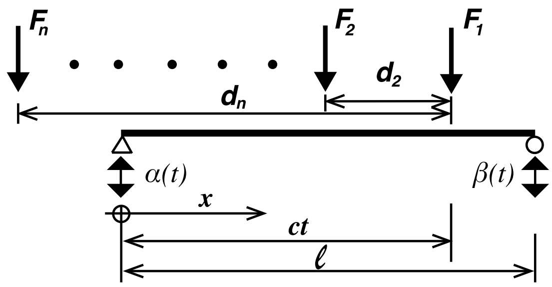


Figure 1: Simplified model of the beam, moving forces, and movement of supports

## 2. Description of the model and closed form solution

Let us assume a simple damped beam of span $\ell$, which is subjected to a row of $n$ moving forces $F_i, i = 1, 2, \dots, n$ at the distances $d_i$, see Figure 1. The forces are moving from the left to the right with a constant velocity $c$. The supports of the beam perform vertical movements $\alpha(t)$ (left support) and $\beta(t)$ (right support), respectively. The problem is governed by the partial differential equation:

$$EI\, v^{\text{IV}}(x,t) + \mu\, \ddot{v}(x,t) + 2\mu\gamma\, \dot{v}(x,t) = \sum_{i=1}^{n} F_i \varepsilon_i(t)\delta(x - d_i), \tag{1}$$

$$v(0,t) = \alpha(t),\ v(\ell,t) = \beta(t), \quad v''(0,t) = 0,\ v''(\ell,t) = 0, \tag{2}$$

$$v(x,0) = \dot{v}(x,0) = 0, \tag{3}$$

where $v(x,t)$ is the vertical displacement of the beam at $x$ and time $t$, respectively, $EI$ is the flexural rigidity of the beam (constant), $\mu$ is the mass per unit length of the beam (constant), $\gamma$ is the circular frequency of the beam damping, $\varepsilon_i(t) = h(t-t_i) - h(t-T_i)$ with $h(t)$ being the Heaviside unit step function, $\delta(x)$ is the Dirac function, $t_i = d_i/c$, $T_i = (\ell + d_i)/c$ is the time when the $i$-th force enters or leaves the beam, $d_i$ is the distance between the first and $i$-th force $d_1 = 0$, and primes and dots denote the differentiation with respect to space and time, respectively.

The boundary conditions (2) characterize the "simply supported beam" with prescribed movement of its both ends. The soil displacement functions are usually assumed to be equal $\alpha(t) = \beta(t)$ or shifted $\alpha(t) = \beta(t \pm \Delta t)$ on both ends, however the general choice $\alpha(t) \neq \beta(t)$ is supposed here.

The closed form solution to the problem of beam vibration (1–3) used in this work is described in detail in [3]. Thus, due to space limitation only a few incomplete formulae will be presented here.

The response of the beam $v(x,t)$ is resolved into the so called quasi-static component $v_s(x,t)$ comprising variable boundary conditions and dynamic component $v_d(x,t)$, which includes the moving load on the right hand side:

$$v(x,t) = v_s(x,t) + v_d(x,t). \tag{4}$$

The time-variable boundary conditions $\alpha(t), \beta(t)$ in equation for $v_s(x,t)$ are assumed to be represented by a sum of $m$ selected (dominant) terms of a finite Fourier approximation, possibly modulated by a function of "slow time" $\tau$, $\tau = \sigma t$, $\sigma \ll 1$,

$$\alpha(t) = \sum\nolimits_{k=1}^{m} \gamma(\tau) \sin \omega_k t. \tag{5}$$

Harmonic character of the boundary conditions and assumption of zero damping enables to find analytical solution as a sum of eigenmodes $v_{s,i}(x,t)$ :

$$v_s(x,t) = \sum\nolimits_{k=1}^{m} v_{s,k}(x,\tau) \sin \omega_k t, \tag{6}$$

$$v_{s,k}(x,\tau) = C_{k,1} \sin \frac{\lambda_k x}{\ell} + C_{k,2} \cos \frac{\lambda_k x}{\ell} + C_{k,3} \sinh \frac{\lambda_k x}{\ell} + C_{k,4} \cosh \frac{\lambda_k x}{\ell}, \tag{7}$$

where $\lambda_k = \ell \left(\mu \omega_k^2 / EI\right)^{\frac{1}{4}}$ and $C_{k,j}(\tau)$ are given by boundary conditions.

The dynamic component can expressed in the form of eigenmodes expansion:

$$v_d(x,t) = \sum\nolimits_{j=1}^{\infty} q_j(t) \sin \frac{j\pi x}{\ell}, \tag{8}$$

where the functions $q_j(t)$ sum contributions of individual forcing components.

## 3. Finite difference schema

Let us assume a uniform discretization of the beam with $N-1$ interior points, $0 = x_0 < x_1 < \ldots < x_N = \ell$, $x_i = ih$. The difference schema for the $4^{\text{th}}$ order derivative in (1) with boundary conditions (2) will be deduced from a transformed system ($z(x,t) = v''(x,t)$ ):

$$\begin{aligned} z''(x,t) + v(x,t) &= f(x,t) \quad \text{and} \quad & v(0,t) &= \alpha(t),\ v(\ell,t) = \beta(t), \\ v''(x,t) - z(x,t) &= 0, & z(0,t) &= 0, \quad z(\ell,t) = 0, \end{aligned} \tag{9}$$

which can be discretized using the standard second order difference schema $h^2 v''(x_i) \approx v(x_{i-1}) - 2v(x_i) + v(x_{i+1})$. This procedure avoids explicit formulation of second order boundary conditions. Eliminating the auxiliary variable $z$ the linear algebraic system conforming to (9) with boundary conditions (2) can be written in the matrix form:

$$\frac{1}{h^4} \mathbf{M} \cdot \mathbf{v}_j = \mathbf{f}_j + \frac{1}{h^4} \mathbf{g}_j. \tag{10}$$

Vector $\mathbf{v}_j$ represents unknown displacements of internal nodes $x_i$, $i = 1, \ldots, N-1$ at time instant $t_j = j \cdot \Delta$. Vector $\mathbf{f}_j = \{f(x_i, t_j)\}_{i=1}^{N-1}$ corresponds to the value of the right hand side in the internal nodes. The symmetric matrix $\mathbf{M} \in \mathbb{R}^{(N-1)\times(N-1)}$ consists of 5 non-zero diagonals with numbers $6, -4, 1$ on the main-, $1^{\text{st}}$, and $2^{\text{nd}}$ sub- and superdiagonal, respectively, with the exception of the corner values: $M_{1,1} = M_{N-1,N-1} = 5$. Elements of vector $\mathbf{g}_j \in \mathbb{R}^{(N-1)}$ are given as

$$\mathbf{g}_j = \left(2\alpha(t), -\alpha(t), 0, \ldots, 0, -\beta(t), 2\beta(t)\right)^T. \tag{11}$$

The time derivatives at $t = t_j = j \cdot \Delta$ will be approximated by formulae

$$\Delta^2 \ddot{v}(t_j) \approx v(t_{j-2}) - 2v(t_{j-1}) + v(t_j)\,, \quad 2\Delta \dot{v}(t_j) \approx v(t_{j-2}) - 4v(t_{j-1}) + 3v(t_j). \quad (12)$$

The final implicit recurrence formula can be written in the matrix form for $j = 1, \ldots$

$$\left(b^2\mathbf{M} + \left(1 + \frac{3}{2}\gamma\Delta\right)\mathbf{I}\right)\cdot\mathbf{v}_j = \frac{\Delta^2}{\mu}\mathbf{f}_j + b^2\mathbf{g}_i + 2(1+\gamma\Delta)\mathbf{v}_{j-1} - \left(1 - \frac{1}{2}\gamma\Delta\right)\mathbf{v}_{j-2}, \quad (13)$$

where

$$b = \sqrt{\frac{EI}{\mu}}\frac{\Delta}{h^2}. \quad (14)$$

In compliance with the initial conditions (3) the two starting values can be considered zero: $\mathbf{v}_{-1} = \mathbf{v}_0 = \mathbf{0}$.

The discretization parameters $h$, $\Delta$ should be chosen to allow consistent description of the moving load. The value of $h$ should correspond to axle distances of the supposed train and the time step $\Delta$ has to be dependent on the train velocity

$$h = \frac{1}{k}\mathrm{GCD}\{d_1, \ldots, d_n\}\,, \quad \Delta t = \frac{1}{l}\frac{h}{c} \qquad \text{for some } \; k, l \in \mathbb{N}. \quad (15)$$

The consistent distribution of the axle load $F_i$ between two adjacent space nodes is necessary if $l > 1$. This can be assured, e.g., by the choice

$$f(x,t) = \sum\nolimits_{i=1}^{n} F_i \max\left\{0, 1 - \left|\frac{x - (ct - d_i)}{h}\right|\right\}. \quad (16)$$

## 4. Numerical verification

Problem 1. The simple benchmark case was used first in [1] and then subsequently several times. It considers free vibration case ($f(x,t) = 0$) of an undamped system (1–2) with parameters $\ell = 1, EI = 1, \mu = 1, \gamma = 0, \alpha(t) = \beta(t) = 0$ and

$$v(x,0) = \frac{1}{12}x(2x^2 - x^3 - 1)\,; \qquad \dot{v}(x,0) = 0 \qquad \text{for } 0 \le x \le 1. \quad (17)$$

The exact solution to the continuous problem is obtained by Fourier series analysis:

$$v(x,t) = \sum\nolimits_{s=1}^{\infty} \frac{4}{s^5\pi^5}\left(\cos(s\pi) - 1\right)\sin(s\pi x)\cos(s^2\pi^2 t). \quad (18)$$

Figure 2(a-b) shows numerical approximation (solid curves) of $v(0.5,t)$ computed using the finite difference recurrence (13) together with the corresponding exact solutions (dashed curves) for two different time steps, $\Delta = 0.005, 0.00125$. The relatively high decrease of the computed amplitude in the plot b) is caused by numerical dispersion (damping), see [4]. The rate of numerical dispersion depends on the value of coefficient $b$ (14). The same coefficient occurs in the stability criterion of explicit difference schemas but with different interpretation.

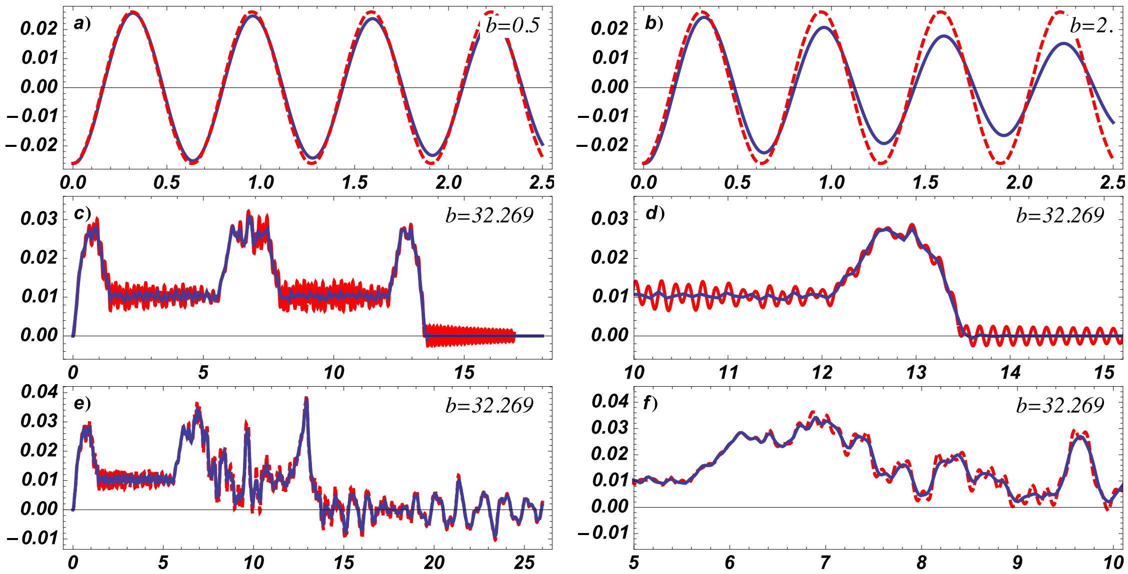


Figure 2: Mid-span deflection ($x = \frac{1}{2}\ell$) of three benchmark cases – numerical approximation (solid, blue) and exact solution (dashed, red).
(a–b) Free vibration benchmark, $h = 0.05$, (a) $\Delta = 0.00125$, (b) $\Delta = 0.005$.
(c–d) Concrete bridge ($\ell = 20$m), train passing at speed $c = 100$ km/h, $h = 0.5, \Delta = 0.0045$, (d) detail for $t \in (10, 15)$.
(e–f) Concrete bridge ($\ell = 20$m), train passing at speed $c = 100$ km/h and an earthquake shock at $t_e = 6.76$s, $\Delta = 0.0045$, (f) detail for $t \in (5, 10)$.
(For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Problem 2. The second example is selected from the parametric study presented by authors in [3]. Parameters of the concrete bridge are specified as $\ell = 20$m, $\mu = 8 \cdot 10^3$kg, $EI = 65.5 \cdot 10^6 \text{m}^3\text{kg.s}^{-2}, \gamma = 1.27\text{s}^{-1}$. The train Talgo AV consists of 2 indentical formations with 7 carriages and 20 axles, 16 tons each. Figure 2(c-d) shows the mid-span deflection of the bridge caused by train cruising at speed of $c$ =100km/h. The three significant peaks are caused by the motorized carriages (one on each end of the train and two in the middle). The highly oscillating curve (red) depicts the approximative analytical solution, only first eigenmode is taken into account. The dark smooth curve corresponds to numerical solution (13) with a relatively large quotient $b \approx 32$. It follows approximately the mean value of the analytical solution. Difference of both solutions in greater detail can be seen in part d) of the figure. The high numerical damping wiped out small oscillations as well as the free vibration after the train left the bridge ($t = 13.5$s).

Problem 3. Figure 2(e-f) shows effect of the combined load of train and earthquake. The earthquake is represented by its several Fourier components and a simple modulation function, see [3]. The shock reaches the bridge at the moment when the first formation of the train leaves the bridge. At this moment is the response due to passing train maximal because the four middle axles forces of the Talgo train represent a pair of engines. It is apparent that after the earthquake shock the amplitude increases. Coincidence between approximate analytical (red, dashed) and numerical (blue, solid) is fairly good: the maximal relative error is $\sim 10\%$ despite the significant simplification of the analytic model and the large time step which leads to $b \approx 32$.

## 5. Conclusions

We presented a simplified analysis of the vertical vibration of a bridge, which is is caused by a concurrent action of a long sequence of axle forces or their groups distributed in almost regular distances and a support motion due to an earthquake. The implicit finite difference scheme was introduced to verify justifiability of the simplifying assumptions of the approximative closed form solution. The computed responses were compared to those obtained using analytical methods with good results: the agreement between analytical and numerical results for the benchmarks was within desired 1% provided that the time step $\Delta$ was sufficiently small. Some problems with high numerical damping are reported.


## Acknowledgements

The kind support of the Czech Science Foundation Project No. GC13-34405J and of the RVO 68378297 institutional support are gratefully acknowledged.


## References


[1] Evans, D. J.: A stable explicit method for the finite-difference solution of a fourth-order parabolic partial-differential equation. Comput. J. **8** (1965), 280–287.

[2] Frýba, L.: *Vibration of solids and structures under moving loads.* 3rd ed., Academia, Prague, Thomas Telford, London, 1999.

[3] Frýba, L., Urushadze, S. and Fischer C.: Vibration of a beam resting on movable supports and subjected to moving loads. In: A. Cunha, E. Caetano, P. Ribeiro, G. Müller (Eds.), *Proceedings of the 9th International Conference on Structural Dynamics, EURODYN 2014*, pp. 1361–1368 (`190_MS06_ABS_1291`). Porto, 2014.

[4] Hoffman, J. D.: *Numerical methods for engineers and scientists.* 2nd ed., Marcel Dekker, Basel, 2001.

[5] Yau, J. D. and Frýba, L.: Response of suspended beams due to moving loads and vertical seismic ground excitations. Eng. Struct. **29** (2007), 3255–3262.