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SMOOTH APPROXIMATION SPACES BASED ON A PERIODIC SYSTEM

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Abstract

A way of data approximation called smooth was introduced by Talmi and Gilat in 1977. Such an approach employs a (possibly infinite) linear combination of smooth basis functions with coefficients obtained as the unique solution of a minimization problem. While the minimization guarantees the smoothness of the approximant and its derivatives, the constraints represent the interpolating or smoothing conditions at nodes. In the contribution, a special attention is paid to the periodic basis system $\exp(-\mathrm{i}kx)$. A 1D numerical example is presented.

1. Introduction

Measurements of the values of a continuous function of one or more independent variables are performed in many branches of science and technology. The data correspond to a finite number of measurement nodes but we need also its extension: the values corresponding to other points in some domain. The way of smooth interpolation [3, 4] is to minimize the L^2 norm of the interpolating function and that of its chosen (possibly all) derivatives. This is a variational problem with constraints represented by the interpolation conditions. An example of a smooth interpolation is the well-known spline interpolation.

We are mostly interested in the case of a single independent variable in the contribution. We generalize the approach of [4], and introduce the problem to be solved and the tools necessary to this aim in Sec. 2. We also quote the general existence theorem for smooth interpolation [3]. We are concerned with the use of basis system $\exp(-\mathrm{i}kx)$ of exponential functions of pure imaginary argument for 1D, 2D, and 3D smooth approximation problems in Sec. 3. In the conclusion, we show and discuss results of numerical experiments to compare the classical interpolation formulae and various kinds of the smooth approximation.

2. Problem of interpolation. Smooth interpolation

Let us have a finite number N of (complex, in general) measured (sampled) values $f_1, f_2, \dots, f_N \in C$ obtained at N mutually distinct nodes $X_1, X_2, \dots, X_N \in R^n$. Assume that $f_j = f(X_j)$ are measured values of some continuous function f. The dimension n of the independent variable may be arbitrary. For the sake of simplicity we put $n = 1$ except for Sec. 3 and assume that $X_1, X_2, \dots, X_N \in \Omega$, where either $\Omega = [a, b]$ is a finite interval or $\Omega = (-\infty, \infty)$.

The *problem of interpolation* is construction of the interpolating function z fulfilling the interpolation conditions

$$z(X_j) = f(X_j), \quad j = 1, \dots, N. \tag{1}$$

The problem of data interpolation does not have a unique solution. The property (1) of the interpolating function is uniquely formulated by mathematical means but there are also additional conditions on the *subjective perception* of the behavior of the interpolating curve between nodes that can hardly be formalized.

An inner product space is introduced to take into account the additional conditions in the problem of smooth interpolation [3], [4]. Let $\{B_l\}_{l=0}^{\infty}$ be a sequence of nonnegative numbers and let L be the smallest nonnegative integer such that $B_L > 0$ while $B_l = 0$ for $l < L$. Let $\widetilde{\mathcal{W}}$ be a linear vector space of complex functions g continuous together with their derivatives of all orders on the interval Ω.

Put

$$(g, h)_L = \sum_{l=0}^{\infty} B_l \int_{\Omega} g^{(l)}(x)[h^{(l)}(x)]^* \, \mathrm{d}x, \qquad |g|_L^2 = \sum_{l=0}^{\infty} B_l \int_{\Omega} |g^{(l)}(x)|^2 \, \mathrm{d}x, \tag{2}$$

where * denotes the complex conjugate.

If $L = 0$ (i.e. $B_0 > 0$), $g \in \widetilde{\mathcal{W}}$, and the value of $|g|_0$ exists and is finite, then $(g, h)_0 = (g, h)$ has the properties of *inner product* and the expression $|g|_0 = \|g\|$ is *norm* in the normed space W_0.

If $L > 0$ let $P_{L-1} \subset \widetilde{\mathcal{W}}$ be the subspace whose basis $\{\varphi_p\}$ consists of monomials $\varphi_p(x) = x^{p-1}$, $p = 1, \dots, L$, and $(\varphi_p, \varphi_q)_L = 0$ for $p \neq q$. Using (2), we construct the *quotient space* $\widetilde{\mathcal{W}}/P_{L-1}$ whose zero class is the subspace P_{L-1}. We see that then $(\cdot, \cdot)_L$ and $|\cdot|_L$ represent the inner product and norm in the normed space $W_L = \widetilde{\mathcal{W}}/P_{L-1}$.

For an arbitrary $L \geq 0$, choose a *basis system* of functions $\{g_k\} \subset W_L$, $k = 1, 2, \dots$, that is complete and orthogonal (in the inner product of W_L), $(g_k, g_m)_L = 0$ for $k \neq m$, $(g_k, g_k)_L = |g_k|_L^2 > 0$. If $L > 0$ then it is, moreover, $(\varphi_p, g_k)_L = 0$ for $p = 1, \dots, L$, $k = 1, 2, \dots$. The set $\{\varphi_p\}$ is empty for $L = 0$.

The *problem of smooth interpolation* consists in finding the coefficients A_k and a_p of the expression $z(x) = \sum_{k=1}^{\infty} A_k g_k(x) + \sum_{p=1}^{L} a_p \varphi_p(x)$ such that (1) holds and the quantity $|z|_L^2$ attains its minimum.

Let the sum $R_L(x,y) = \sum_{k=1}^{\infty} g_k(x)g_k^*(y)|g_k|_L^{-2}$, called the *generating function*, converges for all $x, y \in \Omega$. Theorem 1 of [3] states how to obtain the smooth interpolant z in the form

$$z(x) = \sum_{j=1}^{N} \lambda_j R_L(x, X_j) + \sum_{p=1}^{L} a_p \varphi_p(x), \tag{3}$$

where the coefficients λ_j, $j = 1, \ldots, N$, and a_p, $p = 1, \ldots, L$, are the unique solution of a nonsingular system of $N + L$ linear algebraic equations.

3. A choice of basis function system

Recall that we have put $n = 1$. Let the function f to be approximated be periodic in $[0, 2\pi]$. We choose periodic exponential functions of pure imaginary argument for the basis system $\{g_k\}$. The following theorem shows important properties of the system.

Theorem 1. *Let there be an integer $s \geq L$ such that $B_l = 0$ for all $l > s$ in W_L. The system of periodic exponential functions of pure imaginary argument*

$$g_k(x) = \exp(-\mathrm{i}kx), \quad x \in [0, 2\pi],\ k = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{4}$$

is then complete and orthogonal in W_L.

Proof. The orthogonality and completeness of the system $\{g_k\}$ in $H^s(0, 2\pi)$ is proven, e.g., in [1]. The proof for the space W_L is based on the equivalence of norms. □

The range of k implies a minor change in the notation introduced above. For the basis system (4), notice that

$$R_L(x,y) = \sum_{k=-\infty}^{\infty} \frac{g_k(x)g_k^*(y)}{|g_k|_L^2} = \sum_{k=-\infty}^{\infty} \frac{\exp(-\mathrm{i}k(x-y))}{|g_k|_L^2} \tag{5}$$

is the Fourier series in $L^2(0, 2\pi)$ with the coefficients $|g_k|_L^{-2}$, $|g_k|_L^2 = 2\pi \sum_{l=L}^{\infty} B_l k^{2l}$.

Let now the function f to be approximated be nonperiodic on $(-\infty, \infty)$ and $f^{(l)}(\pm\infty) = 0$ for all $l \geq 0$. Let us define the generating function $R_L(x,y)$ as the Fourier transform of the function $|g_k|_L^{-2}$ of a continuous variable k,

$$R_L(x,y) = \int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}k(x-y))}{|g_k|_L^2}\, \mathrm{d}k, \tag{6}$$

if the integral exists. Using the effect of transition from the Fourier series (5) to the Fourier transform (6), we have transformed the basis functions, enriched their spectrum, and released the requirement of periodicity of f. Moreover, if the integral (6) does not exist, in many instances we can calculate $R_L(x,y)$ as the Fourier transform $\mathcal{F}$ of the generalized function $|g_k|_L^{-2}$ of k.

Choosing now a particular sequence $\{B_l\}$, we complete the definition of the inner product and norm (2) in a particular W_L. Let us thus put $B_l = 0$ for all l with the exception of $B_2 = 1$ (cf. [4]). It means that we have $L = 2$ and minimize the usual L^2 norm of the second derivative of the interpolant (3), $z(x) = \sum_{j=1}^{N} \lambda_j R_2(x, X_j) + a_0 + a_1 x$. We have $|g_k|_2^2 = 2\pi k^4$ and putting $r = |x - y|$, we arrive at

$$R_2(x, y) = \mathcal{F}(1/(2\pi k^4)) = \tfrac{1}{12} r^3, \tag{7}$$

where $\mathcal{F}$ denotes the integral Fourier transform or the Fourier transform of a generalized function [2]. It is easy to find out that this version of smooth approximation is, in fact, the well-known *cubic spline interpolation.*

There are further practical examples of smooth approximation where the integral generating function R_L can be calculated with the help of the Fourier transform.

We can generalize the smooth interpolation procedure of Sec. 2 to R^n, n being a positive integer. We do not introduce the notation in R^n in detail but will you keep in mind that all the derivatives are partial now. We choose the system of periodic exponential functions $g_k(x) = \exp(-\mathrm{i} k \cdot x)$ of pure imaginary vector argument, which can be proven to be complete and orthogonal in W_L, and put r equal to the Euclidean norm of $x - y$.

Let $n = 2$. In the definition of inner product in W_L, we put $L = 2$ and construct analog of a spline in two dimensions. The interpolant has the form $z(x) = \sum_{j=1}^{N} \lambda_j R_2(x, X_j) + a_0 + a_1 x_1 + a_2 x_2$ and it is $|g_k|_2^2 = 2\pi(k_1^2 + k_2^2)^2$. We arrive at $R_2(x, y) = \mathcal{F}(1/(2\pi(k_1^2 + k_2^2)^2)) = C_2 r^2 \ln r + C_2' r^2$, where C_2, C_2' are constants [2].

Let $n = 3$. With the same choice $L = 2$ we construct analog of a spline in three dimensions. The interpolant has the form $z(x) = \sum_{j=1}^{N} \lambda_j R_2(x, X_j) + a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$ and it is $|g_k|_2^2 = 2\pi(k_1^2 + k_2^2 + k_3^2)^2$. We have $R_2(x, y) = \mathcal{F}(1/(2\pi(k_1^2 + k_2^2 + k_3^2)^2)) = C_3 r$, where C_3 is a constant [2].

For $n = 1$, we will also consider another interesting choice of $\{B_l\}$ with the system (4). Putting $L = 0$, $r = |x - y|$ and, in particular, $B_l = D^{2l}/(2l)!$, $D = \frac{1}{3}$, we calculate [4]

$$R_0(x, y) = \frac{1}{2D \cosh(\pi r/(2D))}. \tag{8}$$

4. Computational comparison

To present results of numerical experiments we use two complete and orthogonal systems $\{g_k\}$ in W_L. We assume that the function to be interpolated is not periodic.

(i) Exponential functions of pure imaginary argument (4) {dashed line} with the generating function (7), $L = 2$, $B_2 = 1$, i.e. cubic spline interpolation.

(ii) The same functions (4) {dashed line} with the generating function (8).

(iii) Orthonormalized monomials {dotted line}. The system of monomials $h_k(x) = x^k$, $k = 0, 1, 2, \ldots$, is orthonormalized numerically on $(-1, 1)$ by the Gram-Schmidt procedure with respect to the inner product $(g, h)_0$. We use $L = 0$ and B_l the same as in (ii). $R_0(x, y)$ is evaluated numerically.

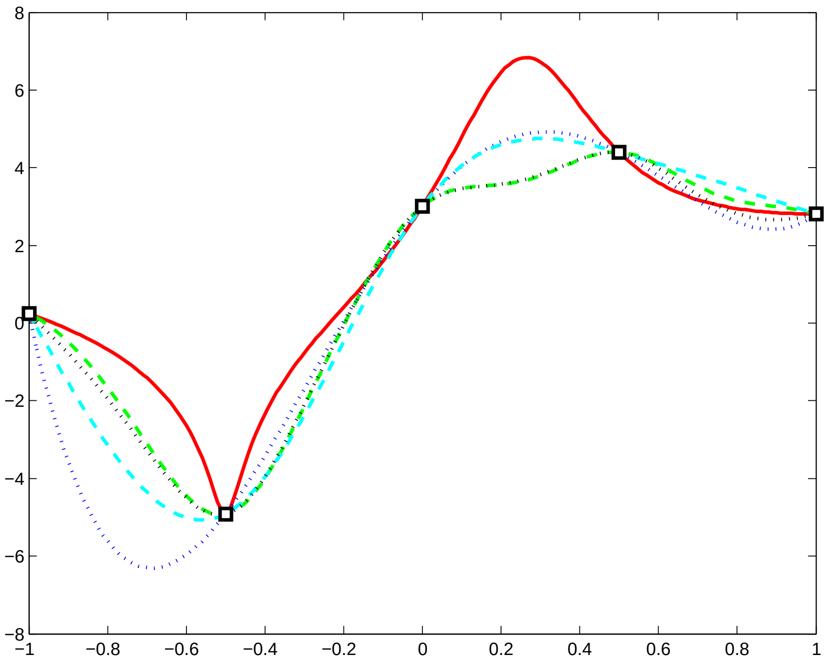

Figure 1: $N = 5$, "pole" $x = 0.25$ is not an interpolation node. Curves at $x = 0.80$ from top to bottom: (i), (ii), true, (iii), (iv)

Next two interpolation methods are classical.

(iv) Polynomial interpolation {dotted line}.

(v) Rational interpolation {dash-dot line}.

The interpolated function

$$f(x) = \ln(\tfrac{1}{100}(x + \tfrac{1}{2})^2 + 10^{-5}) + \frac{6}{1 + 16(x - \frac{1}{4})^2} + 6 \tag{9}$$

has "almost a singularity" at $x = -\frac{1}{2}$ and "almost a pole" at $x = \frac{1}{4}$. The smooth as well as classical interpolation of the function (9) has been constructed in several equidistant grids of N nodes on $[-1, 1]$. Some very inaccurate results (obtained e.g. by the polynomial interpolation of high degree) are omitted in some of the following graphs. In the figures, the solid line represents the true solution, i.e. the function (9). The results of interpolation are in Fig. 1 and 2. They show some qualitative behavior of the results but the quantitative properties can hardly be seen.

5. Conclusion

Since the extent of this contribution is limited we presented only a single example. It would not be fair to draw principal conclusions from it. The computation shows that the smooth interpolation is a competitive method. The L_∞ error of all the methods used, except for error of the polynomial interpolation, decreases as N

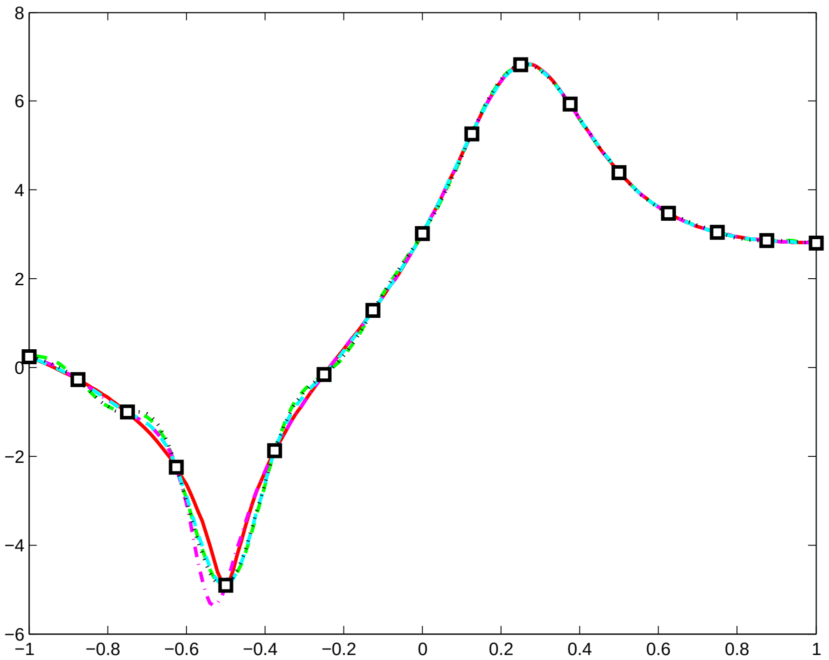

Figure 2: $N = 17$. Curves at $x = -0.55$ from top to bottom: true, (i) identical to (ii) and (iii), (v)

increases. Nevertheless, we should keep in mind that the only ultimate interpolation conditions are the values at nodes.

The case of $n > 1$ is much more interesting and makes many important applications possible. The interpolation nodes can be arbitrarily placed in the plane or space and large sets of data measured can be handled. There are also several further choices of the sequence $\{B_l\}$ that lead to a smooth approximating function possessing some "physical properties" like the cubic spline.

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