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WAVELET BASES FOR THE BIHARMONIC PROBLEM

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Abstract

In our contribution, we study different Riesz wavelet bases in Sobolev spaces based on cubic splines satisfying homogeneous Dirichlet boundary conditions of the second order. These bases are consequently applied to the numerical solution of the biharmonic problem and their quantitative properties are compared.

1. Introduction

Wavelets are an established tool for the numerical solution of operator equations. One of advantages of wavelet methods consists in the existence of a diagonal preconditioner. This preconditioner is optimal in the sense that the condition number of the preconditioned stiffness matrix does not depend on the size of the matrix. Furthermore, a well-known compression property of wavelets enables efficient adaptive solving of operator equations.

In numerical simulations, spline-wavelet bases are of special interest, because they are known in a closed form, they are relatively smooth and they have a small support in comparison with other wavelet bases, e.g. orthonormal wavelet bases. For the numerical treatment of operator equations wavelet bases defined on bounded domain are needed. They are usually derived from wavelet bases on the interval. Recently, several constructions of cubic spline-wavelet bases on the interval adapted to the second order homogeneous Dirichlet boundary conditions were proposed [1, 3, 9, 10]. The bases in [4, 10] have local dual basis functions, which is important in some applications, such as solving nonlinear equations, but for solving partial differential equations the locality of duals is not necessary. Therefore in a construction in [8], the locality of duals is not required. The resulting basis has superb quantitative properties, but wavelets have no vanishing moments. In [5], we also gave up the locality of duals and we designed a cubic spline-wavelet basis with vanishing wavelet moments adapted to homogeneous Dirichlet conditions for the biharmonic problem. In this contribution, we show that our basis have similar excellent quantitative properties as basis from [8] and due to vanishing moments it can be used also in adaptive wavelet methods. In [5], a proof that this basis is a Riesz basis of the space $H_0^s(0,1)$ for $1.5 < s < 2.5$ is presented and properties of the projectors associated with this basis are derived.

2. Construction of wavelet basis

We consider the domain $\Omega \subset \mathbb{R}^d$ and the Sobolev Space $H_0^2(\Omega)$ with the standard $H_0^2(\Omega)$–norm denoted by $\|\cdot\|_{H_0^2(\Omega)}$ and the $H_0^2(\Omega)$–seminorm denoted by $|\cdot|_{H_0^2(\Omega)}$. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j,k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level.* Let

$$l^2(\mathcal{J}) := \left\{ \mathbf{v} : \mathcal{J} \to \mathbb{R}, \sum_{\lambda\in\mathcal{J}} |\mathbf{v}_\lambda|^2 < \infty \right\}, \quad \|\mathbf{v}\|_{l^2(\mathcal{J})} := \left(\sum_{\lambda\in\mathcal{J}} |\mathbf{v}_\lambda|^2 \right)^{1/2}. \tag{1}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *wavelet basis* of $H_0^2(\Omega)$, if

i) Ψ is a *Riesz basis* for $H_0^2(\Omega)$, i.e. the closure of the span of Ψ is $H_0^2(\Omega)$ and there exist constants $c, C \in (0,\infty)$ such that

$$c\,\|\mathrm{b}\|_{l^2(\mathcal{J})} \le \left\| \sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda \right\|_{H_0^2(\Omega)} \le C\,\|\mathrm{b}\|_{l^2(\mathcal{J})}, \quad \mathrm{b} := \{b_\lambda\}_{\lambda\in\mathcal{J}} \in l^2(\mathcal{J}). \tag{2}$$

ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \le \mathrm{C}2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

A wavelet basis is usually formed by two types of functions: scaling functions and wavelets. We focus on a wavelet basis recently constructed in [5] and we briefly review the construction. Let ϕ be a cubic B-spline defined on knots $[0,1,2,3,4]$ and ϕ_b be a cubic B-spline defined on knots $[0,0,1,2,3]$. The graphs of the functions ϕ and ϕ_b are displayed in Figure 1. For $j \in \mathbb{N}$ and $x \in [0,1]$ we set

$$\begin{aligned} \phi_{j,k}(x) &= 2^{j/2}\phi(2^j x - k),\ k = 2, \dots 2^j - 2, \\ \phi_{j,1}(x) &= 2^{j/2}\phi_b(2^j x), \quad \phi_{j,2^j-1}(x) = 2^{j/2}\phi_b(2^j(1-x)). \end{aligned} \tag{3}$$

We define a wavelet ψ as

$$\psi(x) = -\frac{1}{2}\phi(2x) + \phi(2x-1) - \frac{1}{2}\phi(2x-2). \tag{4}$$

Then ψ has two vanishing wavelet moments, i.e.

$$\int_{-\infty}^{\infty} x^k \psi(x)dx = 0, \quad k = 0,1. \tag{5}$$

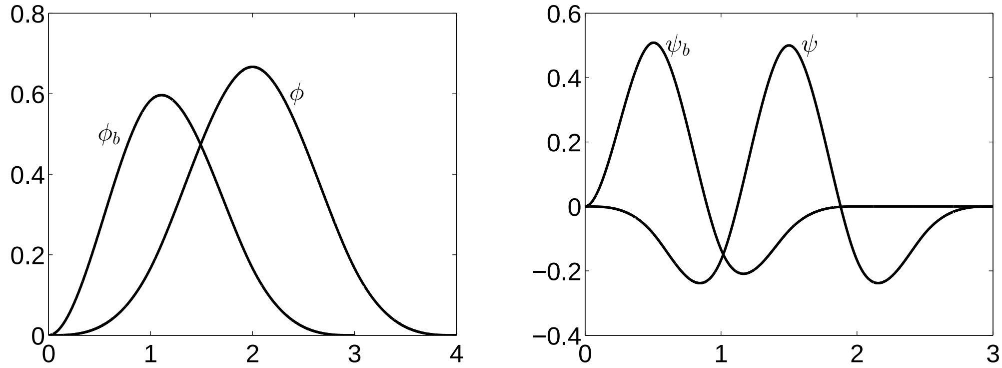

Figure 1: Scaling functions ϕ and ϕ_b and wavelets ψ and ψ_b.

There are several choices for the definition of boundary wavelet. We choose a wavelet with the shortest possible support and the first wavelet moment vanishing:

$$\psi_b(x) = \phi_b(2x) - 0.45\phi(2x). \tag{6}$$

The graphs of the functions ψ and ψ_b are displayed in Figure 1. The inner wavelets correspond to the construction of a wavelet basis for the space $L^2(\mathbb{R})$ in [7].

For $j \in \mathbb{N}$ and $x \in [0,1]$ we define

$$\begin{aligned} \psi_{j,k}(x) &= 2^{j/2}\psi(2^j x - k + 2), k = 2, ..., 2^j - 1, \\ \psi_{j,1}(x) &= 2^{j/2}\psi_b(2^j x), \quad \psi_{j,2^j}(x) = 2^{j/2}\psi_b(2^j(1-x)). \end{aligned} \tag{7}$$

We denote

$$\begin{aligned} \Phi_j &= \left\{\phi_{j,k}/\left|\phi_{j,k}\right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j - 1\right\}, \\ \Psi_j &= \left\{\psi_{j,k}/\left|\psi_{j,k}\right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j\right\}. \end{aligned} \tag{8}$$

Then the sets

$$\Psi_s = \Phi_2 \cup \bigcup_{j=2}^{1+s} \Psi_j \quad \text{and} \quad \Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j \tag{9}$$

are a multi-scale wavelet basis and a wavelet basis of the space $H_0^2(0,1)$, respectively. We use $u \otimes v$ to denote the tensor product of functions u and v, i.e. $u \otimes v(x_1, x_2) = u(x_1) v(x_2)$. We set

$$\begin{aligned} F_j &= \left\{\phi_{j,k} \otimes \phi_{j,l} \,/\, \left|\phi_{j,k} \otimes \phi_{j,l}\right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j - 1\right\} \\ G_j^1 &= \left\{\phi_{j,k} \otimes \psi_{j,l} \,/\, \left|\phi_{j,k} \otimes \psi_{j,l}\right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j - 1, l = 1, \ldots 2^j\right\} \\ G_j^2 &= \left\{\psi_{j,k} \otimes \phi_{j,l} \,/\, \left|\psi_{j,k} \otimes \phi_{j,l}\right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j, l = 1, \ldots 2^j - 1\right\} \\ G_j^3 &= \left\{\psi_{j,k} \otimes \psi_{j,l} \,/\, \left|\psi_{j,k} \otimes \psi_{j,l}\right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j\right\} \end{aligned}$$

where $\Omega = [0,1]^2$. A wavelet basis and a multi-scale wavelet basis of the space $H_0^2(\Omega)$ are defined as

$$\Psi_s^{2D} = F_2 \cup \bigcup_{j=2}^{1+s} \left(G_j^1 \cup G_j^2 \cup G_j^3\right), \quad \Psi^{2D} = F_2 \cup \bigcup_{j=2}^{\infty} \left(G_j^1 \cup G_j^2 \cup G_j^3\right). \tag{10}$$

3. Condition numbers of stiffness matrices

In this section, we compare the condition numbers of the stiffness matrices for the biharmonic problem in two dimensions for different wavelet bases. We consider the biharmonic equation

$$\Delta^2 u = f \quad \text{on } \Omega = (0,1)^d, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \tag{11}$$

Let $\langle \cdot, \cdot \rangle$ denote the standard $L^2(\Omega)$–inner product and Ψ^d be a wavelet basis of $H_0^2(\Omega)$. The variational formulation is $\mathbf{Au} = \mathbf{f}$, where $\mathbf{A} = \left\langle \Delta\Psi^d, \Delta\Psi^d \right\rangle$, $u = \mathbf{u}^T \Psi^d$, and $\mathbf{f} = \left\langle f, \Psi^d \right\rangle$. It is known that then $\operatorname{cond} \mathbf{A} \leq C < \infty$. Since $\mathbf{A}$ is symmetric and positive definite, we have also

$$\operatorname{cond} \mathbf{A}_s \leq C, \qquad \text{where } \mathbf{A}_s = \left\langle \Delta\Psi_s^d, \Delta\Psi_s^d \right\rangle \tag{12}$$

and Ψ_s^d is a multiscale wavelet basis with s levels of wavelets. The condition numbers of the stiffness matrices $\mathbf{A}_s$ are shown in Table 1. A construction by Jia and Zhao from [8] is denoted as JZ11, a construction from [4] is denoted as CF12, a construction of multiwavelet basis from [10] is denoted as S09 and a wavelet basis defined in Section 2 is denoted as new.

s	N	JZ11	N	CF12	N	S09	N	new
				1D				
1	15	45.9	17	61.2	30	472.0	7	3.5
5	255	45.9	257	66.6	510	640.8	127	4.1
9	4095	45.9	4097	66.7	8190	731.4	2047	4.1
				2D				
1	225	34.0	289	128.1	900	484.4	49	8.5
2	961	34.9	1089	141.3	3844	583.4	225	14.3
3	3969	35.1	4225	212.0	15876	626.9	961	17.5
4	16129	35.3	16641	257.6	64516	653.5	3969	18.2
5	65025	35.5	66049	281.2	260100	673.2	16129	18.4
6	261121	35.8	263169	297.2	1044484	689.4	65025	18.6

Table 1: The condition numbers of the stiffness matrices $\mathbf{A}_s$ of the size $N \times N$ corresponding to multi-scale wavelet bases with s levels of wavelets.

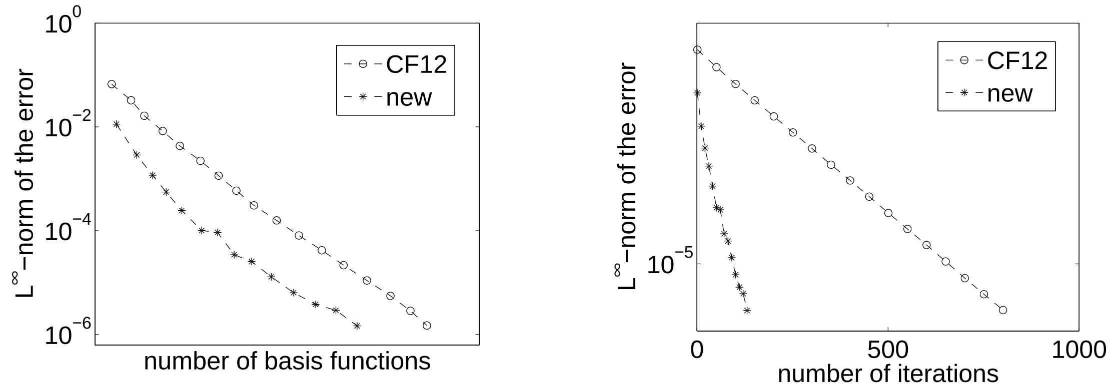

Figure 2: The convergence history for an adaptive wavelet scheme with various wavelet bases.

4. Numerical example

We compare the quantitative behaviour of the adaptive wavelet method with a basis constructed in this paper and a cubic spline-wavelet basis from [4]. In [4] the comparison with other wavelet bases is already done. We consider the equation (11) with a solution u given by

$$u\,(x,y) = v\,(x)\,v\,(y)\,, \quad v\,(x) = x^2\left(1 - e^{12x-12}\right)^2. \tag{13}$$

The solution exhibits a sharp gradient near the point $[1,1]$. We solve the problem by the method designed in [6] with the approximate multiplication of the stiffness matrix with a vector proposed in [2]. The convergence history is shown in Figure 2. In our experiments, the convergence rate, i.e. the slope of the curve, is similar for both bases. However, they significantly differ in the number of basis functions and number of iterations needed to resolve the problem with desired accuracy.

5. Conclusion

We have shown that a wavelet basis from [5] has a short support and the condition number of the corresponding stiffness matrix is smaller than for any other cubic spline wavelet basis adapted to the second-order homogeneous Dirichlet boundary conditions known from literature. It was shown in [8] that Galerkin wavelet method with the wavelet basis from [8] has superb convergence. We have shown that our basis has similar quantitative properties as basis constructed by Jia and Zhao and additionally wavelets have some vanishing wavelet moments. Therefore, unlike basis by Jia and Zhao our basis can be used in adaptive wavelet methods. We implemented adaptive wavelet method with our basis and we have shown that its convergence is improved. However, our basis does not have local duals, therefore in some applications bases from [4, 10] are more appropriate. Furthermore, it should be shown that our basis is indeed a wavelet basis, i.e. that a Riesz basis property (2) is satisfied. The proof and other details can be found in [5].

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References

[1] Černá, D. and Finěk, V.: Cubic spline wavelets satisfying homogeneous boundary conditions for fourth order problems. In: T. E. Simos et al. (Eds.), *Numerical analysis and advanced applications*, AIP Conference Proceedings, vol. 1168, pp. 77–80. American Institute of Physics, New York, 2009.

[2] Černá, D. and Finěk, V.: Approximate multiplication in adaptive wavelet methods. Accepted for publication in Cent. Eur. J. Math., 2012.

[3] Černá, D. and Finěk, V.: Construction of optimally conditioned cubic spline wavelets on the interval. Adv. Comput. Math. **34** (2011), 519–552.

[4] Černá, D. and Finěk, V.: Cubic spline wavelets with complementary boundary conditions. Appl. Math. Comput. **219** (2012), 1853–1865.

[5] Černá, D. and Finěk, V.: Cubic spline wavelets with short support for fourth-order problems. In preparation.

[6] Cohen, A., Dahmen, W. and DeVore, R.: Adaptive wavelet methods II - beyond the elliptic case. Found. Math. **2** (2002), 203–245.

[7] Han, B. and Shen, Z.: Wavelets with short support. SIAM J. Math. Anal. **38** (2006), 530–556.

[8] Jia, R.Q. and Zhao, W.: Riesz bases of wavelets and applications to numerical solution of elliptic equations. Math. Comput. **80** (2011), 1525–1556.

[9] Primbs, M.: New stable biorthogonal spline-wavelets on the interval. Result. Math. **57** (2010), 121–162.

[10] Schneider, A.: Biorthogonal cubic Hermite spline multiwavelets on the interval with complementary boundary conditions. Result. Math. **53** (2009), 407–416.