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ON THE OPTIMAL SETTING OF THE *hp*-VERSION OF THE FINITE ELEMENT METHOD

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Abstract

The goal of this contribution is to find the optimal finite element space for solving a particular boundary value problem in one spatial dimension. In other words, the optimal use of available degrees of freedom is sought after. This is done through optimizing both the mesh and the polynomial degree of the basis functions. The resulting combinatorial optimization problem is solved in parallel by a Matlab program running on a cluster of multi-core personal computers.

1. Introduction

A finite element mesh is among principal factors that affect the performance of the h-version of the finite element method (FEM). An appropriately defined mesh or, to be more correct, a sequence of appropriately defined meshes can accelerate the convergence of the method. Since the FEM projects the exact solution to the mesh-dependent finite element space, the distance between the exact solution and the finite element (FE) space determines the error, that is, the distance between the exact solution and its FE approximation. Various techniques have been proposed to adaptively modify FE meshes and, consequently, FE spaces in order to minimize the error [2, 3, 10].

In the h-version of the FEM, however, the polynomials forming the basis of the FE space either remain unchanged during the mesh modification process or only limited increase/decrease of the polynomial degree is allowed. Typically, piecewise linear and quadratic or even cubic functions are considered.

In the hp-version of the FEM, both mesh and polynomial degree modifications are supported and low as well as higher order polynomials can be found together in FE spaces, see [5, 6, 8, 11, 12]. Nevertheless, this freedom has its dark side. Unlike the h-version of the FEM, where the FE space improvement is mediated solely by adaptive mesh optimization, the mesh as well as the polynomial degree can be adaptively changed in the hp-FEM and it is difficult to determine which of the two approaches is more efficient or how to combine them to get best results. We refer to [1, 7, 9, 13] for various algorithms and analyses focusing on one-dimensional boundary value problems (BVPs).

This contribution presents computational results of the optimization of FE spaces that have a fixed dimension. The goal of the optimization is to minimize the difference between a FE solution and the exact solution of a BVP. The difference is measured in the H^1-norm. The results can (a) serve as benchmarks for the performance of adaptive algorithms, and (b) help to evaluate the efficiency of polynomial degree optimization and mesh optimization.

2. Optimization problem

Let $u(x) = 1/(1.25 - x)$ and let f, a, and b be inferred to comply with the following BVP on the interval $[-1, 1]$

$$-u'' + u = f, \tag{1}$$

$$u'(-1) = a, \quad u'(1) = b. \tag{2}$$

Omitting the knowledge of u, we solve (1)–(2) by the FEM: Find $u_{\mathcal{T}_h,p} \in V^{\mathcal{T}_h,p}$ such that

$$\int_{-1}^{1} \left(u'_{\mathcal{T}_h,p} v'_{\mathcal{T}_h,p} + u_{\mathcal{T}_h,p} v_{\mathcal{T}_h,p}\right) \mathrm{d}x = \int_{-1}^{1} f v_{\mathcal{T}_h,p} \,\mathrm{d}x + b v_{\mathcal{T}_h,p}(1) - a v_{\mathcal{T}_h,p}(-1) \tag{3}$$

holds for any $v_{\mathcal{T}_h,p} \in V^{\mathcal{T}_h,p}$. The finite element space $V^{\mathcal{T}_h,p}$ is defined on the mesh $\mathcal{T}_h$ determined by points $-1 = x_0 < x_1 < \cdots < x_m = 1$. If $C([-1, 1])$ denotes the space of continuous functions on $[-1, 1]$ and $P_{d_k}([x_{k-1}, x_k])$ is the space of polynomials on $[x_{k-1}, x_k]$ of degree d_k or less, we have

$$V^{\mathcal{T}_h,p} = \left\{ v_{\mathcal{T}_h,p} \in C([-1,1]) : \; v_{\mathcal{T}_h,p}|_{[x_{k-1},x_k]} \in P_{d_k}([x_{k-1}, x_k]),\; k = 1, \ldots, m \right\}.$$

The basis functions of $V^{\mathcal{T}_h,p}$ are defined via the Lobatto shape functions (LSFs; see [12]) with their polynomial degree limited to at most 10. Let us note that each LSF of order two and higher is a bubble function because its support comprises only one mesh subinterval.

Various FE spaces can be designed with the same dimension N. To this end, we introduce $p = (d_1, \ldots, d_m)$, m-tuples that describe the polynomial degree distribution over the mesh intervals. By counting the LSFs inclusive of piecewise linear basis functions, we arrive at $N = d_1 + \cdots + d_m + 1$.

Next, let $\mathcal{P}_N$ be the set of all polynomial degree distributions that correspond to N-dimensional FE spaces. As an example, take $N = 5$ and

$$\mathcal{P}_5 = \{(1, 1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2), (2, 2), (3, 1), (1, 3), (4)\},$$

where $(1, 1, 1, 1)$ represents a FE space with five piecewise linear functions and three unspecified mesh nodes between -1 and 1 (inner nodes), whereas (4) represents the unique FE space formed by quartic polynomials on $[-1, 1]$.

Each $p \in \mathcal{P}_N$ determines a family $\mathcal{M}_p$ of meshes $\mathcal{T}_h$ that, if combined with the polynomial degree distribution p, lead to FE spaces with the dimension N.

As already indicated, we are interested in the minimization of

$$\Phi(p, \mathcal{T}_h) = \|u - u_{\mathcal{T}_h,p}\|_{H^1(-1,1)}$$

where $u_{\mathcal{T}_h,p} \in V^{\mathcal{T}_h,p}$ solves (3). More precisely, if a fixed dimension N is given, we search for $p^0 \in \mathcal{P}_N$ and $\mathcal{T}_h^0$ such that

$$\Phi(p^0, \mathcal{T}_h^0) = \min_{p \in \mathcal{P}_N} \min_{\mathcal{T}_h \in \mathcal{M}_p} \Phi(p, \mathcal{T}_h). \tag{4}$$

Problem (4) was solved in the MATLAB® environment. To avoid mesh degeneration, a minimum distance of mesh nodes was bounded from below by a small positive constant.

The position of mesh nodes was optimized by the MATLAB® Optimization Toolbox™ `fmincon` function designed to search for local minima. Since the goal of the inner minimization in (4) is to find a global minimum, multiple runs of `fmincon` were performed on an initial uniform mesh as well as on a number of initial random meshes.

The computational complexity of problem (4) is rapidly increasing with N. Indeed, $|\mathcal{P}_N|$, the cardinality of $\mathcal{P}_N$, is equal to 2^{N-2} if $N = 3, 4, \ldots, 11$. For $N > 11$, the constraint put on the maximum polynomial degree inhibits the exponential growth of $|\mathcal{P}_N|$, but not strongly. It is $|\mathcal{P}_{14}| = 4088$, for instance.

The inner minimizations are mutually independent for different $p \in \mathcal{P}_N$ and were solved in parallel on a cluster of personal computers with (up to) 200 cores.

3. Results

Let $N = 14$. Figure 1 (left) shows the values $\Phi(p, \mathcal{T}_h)$ where $\mathcal{T}_h$ are uniform (non-optimized) meshes. The numbers on the horizontal axis correspond to the position of a particular p in the sequence of all $p \in \mathcal{P}_{14}$. The dependence of p on its ordinal number cannot be given by a simple formula. Let us only say that, very roughly, the higher the ordinal number, the higher the polynomial degrees in p.

We observe that $\Phi(p, \mathcal{T}_h)$ is rather sensitive to p because the values span from 0.0062 (minimum, $p = (3, 10)$) to 2.895 (maximum, $p = (6, 6, 1)$ or $p = (10, 2, 1)$, for example).

The right part of Figure 1 depicts the histogram of $\Phi(p, \mathcal{T}_h)$ on uniform meshes.

Figure 2 is an analogy to Figure 1; it presents the same type of results for optimized meshes. The dependence on p is clearly visible. The S-shaped patterns correspond to the structure of the ordering of $\mathcal{P}_{14}$. In each pattern, $\Phi(p, \mathcal{T}_h)$ decreases if the higher order polynomials move towards the right-end of the mesh. The first pattern from the left begins with $p = (1, 1, \ldots, 1)$ giving the maximum $\Phi(p, \mathcal{T}_h) = 0.298$ and ends with $p = (1, 2, 2, \ldots, 2)$ and $\Phi(p, \mathcal{T}_h) = 0.075$; ordinal

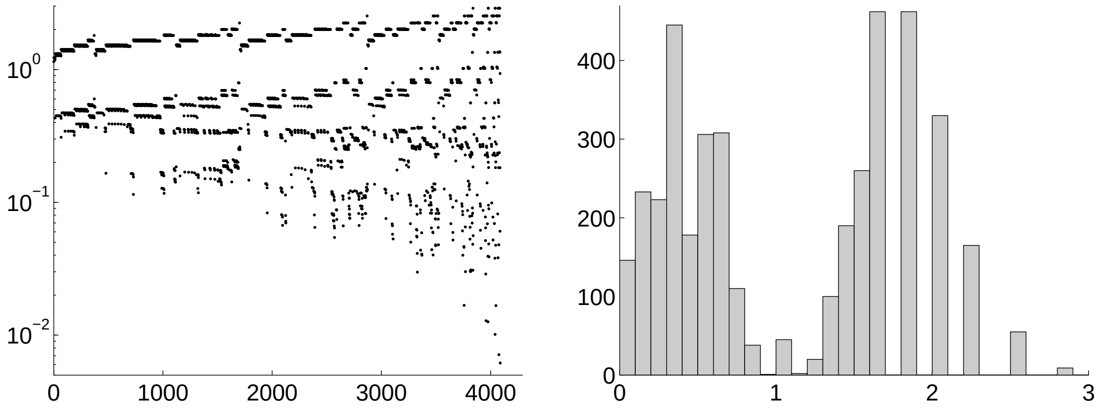

Figure 1: $N = 14$, uniform meshes. Values $\Phi(p, \mathcal{T}_h)$ (left) and the histogram (right).

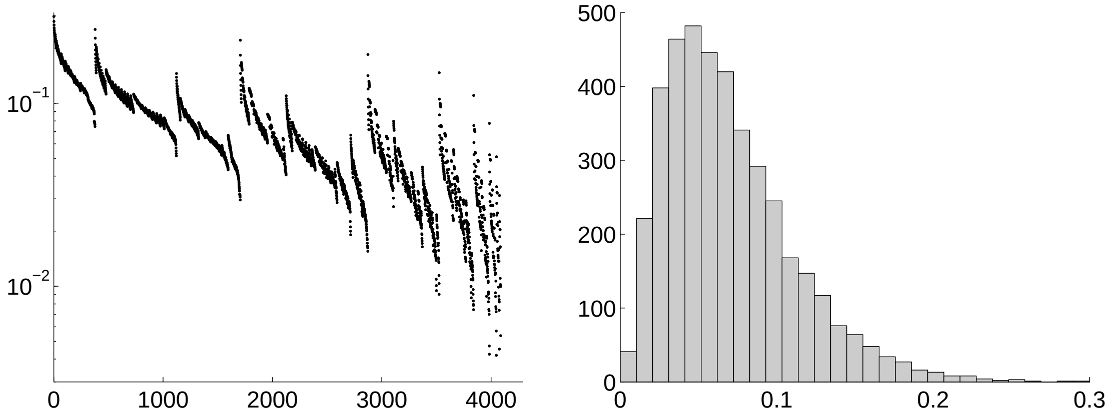

Figure 2: $N = 14$, optimal meshes. Values $\Phi(p, \mathcal{T}_h)$ (left) and the histogram (right).

number 377. The next pattern begins with $p = (3, 1, 1, \dots, 1)$ and $\Phi(p, \mathcal{T}_h) = 0.253$; ordinal number 378.

By comparing the cluster of minimum and near-to-minimum values in Figure 1 and Figure 2, we also infer that though the exact solution u is not a polynomial, it is sufficiently well approximated by a few higher order polynomials. The minimum value of $\Phi(p, \mathcal{T}_h)$ attained on the optimized meshes is equal to 0.0042 if $p = (5, 8)$. This is not a significant improvement over the uniform meshes.

Although the sensitivity to p is strong in the optimal mesh results, we should not overlook the decrease in Φ. Even for the worst-case p, the error is one order lower if the mesh is optimal. This is not the only evidence that mesh optimization

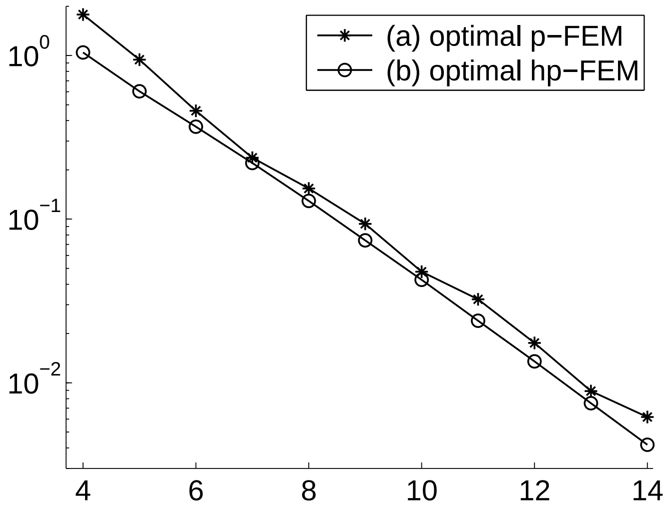

Figure 3: Convergence of the minimum values of Φ if (a) p is optimal and $\mathcal{T}_h$ is uniform; (b) both p and $\mathcal{T}_h$ are optimal. The horizontal axis shows N and the vertical axis shows Φ, the error.

pays off. Let us compare the histograms. Among uniform meshes, only 146 degree distributions guarantee the error less than 0.1; see the first bar in Figure 1 (right). For the optimized meshes, we obtain more than 3300 such degree distributions.

Figure 3 shows the rate of convergence of both optimal p-FEM and optimal hp-FEM. If evaluated through the minimum values of Φ, the difference between the two methods applied to (3) is small. However, one should take into account that there are only a few optimal and almost optimal p distributions on uniform meshes, but significantly more p-$\mathcal{T}_h$ couples can guarantee good performance if the mesh is optimized; consider $0 < \Phi(p, \mathcal{T}_h) \leq 0.05$ and compare Figure 1 and Figure 2. As a consequence, although we strive to optimize both the mesh and p in the hp-FEM, it seems to be advisable to pay somewhat more attention to the former than to the latter. This conclusion agrees with that of [4] where a more detailed analysis of a different BVP is presented.

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