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SUPERAPPROXIMATION OF THE PARTIAL DERIVATIVES IN THE SPACE OF LINEAR TRIANGULAR AND BILINEAR QUADRILATERAL FINITE ELEMENTS

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Abstract

A method for the second-order approximation of the values of partial derivatives of an arbitrary smooth function $u = u(x_1, x_2)$ in the vertices of a conformal and nonobtuse regular triangulation $\mathcal{T}_h$ consisting of triangles and convex quadrilaterals is described and its accuracy is illustrated numerically. The method assumes that the interpolant $\Pi_h(u)$ in the finite element space of the linear triangular and bilinear quadrilateral finite elements from $\mathcal{T}_h$ is known only.

1. Introduction

The problem to find second-order approximations of the first partial derivatives of smooth functions u in the vertices of triangulations by means of the interpolant $\Pi_h(u)$ only is actual since its formulation in [6] in the year 1967. Besides the widely acknowledged method [7] there exist successful methods like [5] and [3]. In this paper, we generalize the method of averaging from [2] to nonobtuse regular triangulations consisting of triangles as well as convex quadrilaterals in general. Numerical experiments indicate the second-order accuracy of this procedure. These high-order approximations of the partial derivatives have many applications. See [1] for some of them.

We denote $[a_1, a_2]$ the Cartesian coordinates of a point a and $|ab|$ the length of the segment $\overline{ab}$. For arbitrary points $a^1, \ldots, a^m$, operations ,,+" and ,,−" mean addition and subtraction modulo m on the set $\{1, \ldots, m\}$.

2. Bilinear quadrilateral finite elements

Besides the linear triangular finite elements, we work with the following bilinear quadrilateral ones.

Definition 1. A *reference bilinear finite element* consists of

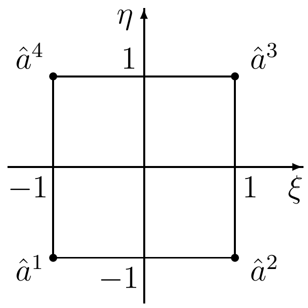

Figure 1: The reference square.

a) the *reference square* $\hat{K} = \overline{\hat{a}^1\hat{a}^2\hat{a}^3\hat{a}^4}$ from Fig. 1,

b) the *local space* $\mathbb{Q}^{(1)} = \{a + b\xi + c\eta + d\xi\eta \mid a, b, c, d \in \mathbb{R}\}$ and of

c) the *parameters* $\hat{p}(\hat{a}^1), \ldots, \hat{p}(\hat{a}^4)$ related to every function $\hat{p} \in \mathbb{Q}^{(1)}$. The parameters determine the function $\hat{p}$ uniquely.

Definition 2. A *bilinear quadrilateral finite element* consists of

a) an image $K = \overline{a^1a^2a^3a^4}$ of $\hat{K}$ by the injective bilinear mapping

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = F_K(\xi, \eta) \equiv \sum_{i=1}^{4} \hat{N}^i(\xi, \eta) \begin{bmatrix} a_1^i \\ a_2^i \end{bmatrix} \tag{1}$$

with the *Lagrange base functions*

$$\hat{N}^1(\xi, \eta) = (1 - \xi)(1 - \eta)/4, \quad \hat{N}^2(\xi, \eta) = (1 + \xi)(1 - \eta)/4,$$
$$\hat{N}^3(\xi, \eta) = (1 + \xi)(1 + \eta)/4, \quad \hat{N}^4(\xi, \eta) = (1 - \xi)(1 + \eta)/4$$

in the space $\mathbb{Q}^{(1)}$ related to the nodes $\hat{a}^1, \ldots, \hat{a}^4$ consecutively. Then $F_K(\hat{a}^i) = a^i$ for $i = 1, \ldots, 4$ obviously and F_K is an injection if and only if K is a *convex quadrilateral*, i.e. the inner angle $\angle a^{i-1}a^ia^{i+1}$ of K is less than π for $i = 1, \ldots, 4$ due to [4], Section 3.3,

b) the *local space* $\mathbb{Q}_K^{(1)} = \{q \mid q = \hat{q} \circ F_K^{-1} \text{ for some } \hat{q} \in \mathbb{Q}^{(1)}\}$ and of

c) the *parameters* $q(a^1), \ldots, q(a^4)$ related to every $q \in \mathbb{Q}_K^{(1)}$. The parameters determine the function q uniquely.

Lemma 1. *The functions* $1, x_1, x_2$ *belong to* $\mathbb{Q}_K^{(1)}$ *for every convex quadrilateral* K.

Proof. If $K = \overline{a^1a^2a^3a^4}$ is a convex quadrilateral then $\mathbb{Q}_K^{(1)} = \{q \mid q \circ F_K \in \mathbb{Q}^{(1)}\}$ is a direct consequence of Definition 2. This and

$$\begin{aligned} 1 \circ F_K &= 1 \in \mathbb{Q}^{(1)} \\ x_1 \circ F_K &= \hat{N}^1(\xi, \eta)a_1^1 + \ldots + \hat{N}^4(\xi, \eta)a_1^4 \in \mathbb{Q}^{(1)} \\ x_2 \circ F_K &= \hat{N}^1(\xi, \eta)a_2^1 + \ldots + \hat{N}^4(\xi, \eta)a_2^4 \in \mathbb{Q}^{(1)} \end{aligned}$$

give us the statement.

Definition 3. If K is a triangle and convex quadrilateral then we denote by $\Pi_K(u)$ the linear and bilinear interpolant of a function $u \in C(K)$ in the vertices of K, respectively.

Lemma 2. *Let us consider a bilinear quadrilateral finite element* $K = \overline{a^1 a^2 a^3 a^4}$, $l = 1, 2$ *and a linear triangular finite element* $T_j = \overline{a^{j-1} a^j a^{j+1}}$. *Then the graph of* $\Pi_{T_j}(u)$ *is the tangent plane to that of* $\Pi_K(u)$ *at the point* a^j, *so that*

$$\frac{\partial \Pi_K(u)}{\partial x_l}(a^j) = \frac{\partial \Pi_{T_j}(u)}{\partial x_l} \quad \forall\, u \in C(K)$$

for $j = 1, \ldots, 4$.

Proof. As the functions from $\mathbb{Q}_K^{(1)}$ are linear on every side of K, $\Pi_K(u)$ is linear on the segments $\overline{a^{j-1} a^j}$ and $\overline{a^j a^{j+1}}$. Hence the segments $\overline{p^{j-1} p^j}$ and $\overline{p^j p^{j+1}}$ for $p^i = [a_1^i, a_2^i, u(a^i)]$, $i = j-1, j, j+1$, are subsets of graph$(\Pi_K(u))$. These segments belong to a unique plane. This one is the tangent plane of graph$(\Pi_K(u))$ at a^j and it contains graph$\big(\Pi_{T_j}(u)\big)$ as well. Lemma 2 follows immediately.

3. Nonobtuse regular triangulations

The symbols $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ are reserved for the spaces of real linear and quadratic polynomials in two variables and Ω for a non-empty bounded connected polygonal domain in the plane. We say that K is an *element* when K is a triangle or a convex quadrilateral, denote $|K|$ the area of K, h_K the diameter of K and ϱ_K the maximal diameter of the circles inside of K.

A system $\mathcal{T}_h$ of elements is said to be a *triangulation* of Ω when $\cup_{K \in \mathcal{T}_h} K = \overline{\Omega}$, any two different elements have disjoint interiors and any side of an element is either a side of another element or a subset of the boundary $\partial\Omega$. Let us consider a *vertex* a of (an element from) a triangulation $\mathcal{T}_h$. We call b a *neighbour* of a (in $\mathcal{T}_h$) when the segment $\overline{ab}$ is a side of an element from $\mathcal{T}_h$ and denote $\mathcal{N}_h(a)$ the set of neighbours of a in $\mathcal{T}_h$. We say that a is an *inner* and *boundary* vertex when $a \in \Omega$ and $a \in \partial\Omega$, respectively.

Definition 4. A system $\mathbf{T}$ of triangulations of Ω is said to be

a) a *family* when for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathbf{T}$ satisfying $h_K < \varepsilon$ for all $K \in \mathcal{T}_h$.

b) *shape-regular* when there is $\sigma > 0$ such that $\varrho_K / h_K > \sigma$ for all elements K of any triangulation from $\mathbf{T}$.

We work with a shape-regular family $\mathbf{T}$ of triangulations of Ω such that all inner angles of the triangles from any triangulation in $\mathbf{T}$ are less than or equal to the right angle. We call these triangulations *nonobtuse regular*.

4. The method of averaging

It is well-known that $\partial u/\partial x_l(a) = \partial \Pi_K(u)/\partial x_l(a) + O(h_K)$ for a vertex a of an element K from a nonobtuse regular triangulation, function $u \in C^2(K)$ and for $l = 1, 2$. We construct a weight vector such that the corresponding weighted average of the values of $\partial \Pi_K(u)/\partial x_l$ in various vertices of the elements K with vertex a approximates $\partial u/\partial x_l(a)$ with an error of the second order. A special case of this construction has been analysed in [2] for the nonobtuse regular triangulations consisting of triangles only.

Calculating the approximations of $\partial u/\partial x_l(a)$, we use local Cartesian coordinates with origin a.

Defrinition 5. Let $\mathcal{T}_h$ be a nonobtuse regular triangulation. We say that $r = (b^1, \ldots, b^n)$ is a *ring* around

a) an inner vertex a of $\mathcal{T}_h$ when

a1) $\{b^1, \ldots, b^n\} \supseteq \mathcal{N}_h(a)$ and

$$b^i \notin \mathcal{N}_h(a) \implies K = \overline{ab^{i-1}b^ib^{i+1}} \in \mathcal{T}_h \text{ and } \angle b^{i-1}ab^{i+1} > \pi/2,$$

a2) $\angle b^nab^1, \ldots, \angle b^{n-1}ab^n$ have the same orientation and

a3) $\angle b^nab^1 + \cdots + \angle b^{n-1}ab^n = 2\pi$.

b) a boundary vertex a of $\mathcal{T}_h$ when there is an inner vertex b^j such that

b1) $(b^1, \ldots, b^{j-1}, a, b^{j+1}, \ldots, b^n)$ is a ring around b^j with $n \geq 5$ or

b2) $\overline{ab^{j+1}b^jb^{j-1}} \in \mathcal{T}_h$ and $(b^1, \ldots, b^{j-1}, b^{j+1}, \ldots, b^n)$ is a ring around b^j.

We say that the triangles $U_1 = \overline{b^nab^1}, \ldots, U_n = \overline{b^{n-1}ab^n}$ are *related* to r and set $H(a) = \max_{1 \leq i \leq n} |ab^i|$.

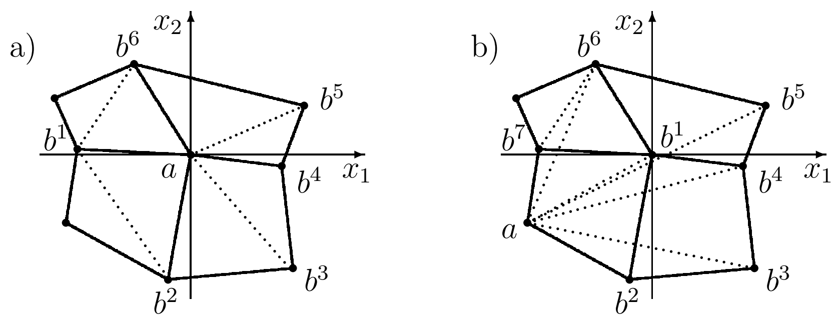

Figure 2: A ring around a) an inner vertex a and b) a boundary one.

In Fig. 2, the thick lines denote the quadrilaterals from the given triangulation and the dotted lines indicate triangles $U_1, \ldots, U_6$ in the case a) and $U_1, \ldots, U_7$ in b).

Definition 6. Let $l = 1, 2$, $r = (b^1, \ldots, b^n)$ be a ring around a vertex a of a nonobtuse regular triangulation and let $u \in C(\overline{\Omega})$. Then we set

$$\mathrm{B}_l[u](a) = f_1 \frac{\partial \Pi_1(u)}{\partial x_l} + \cdots + f_n \frac{\partial \Pi_n(u)}{\partial x_l}. \tag{2}$$

Here $\Pi_1(u), \ldots, \Pi_n(u)$ are the linear interpolants of u in the vertices of the triangles $U_1, \ldots, U_n$ related to r and the *weight vector* $f = [f_1, \ldots, f_n]^\top$ is the minimal 2-norm vector such that $\mathrm{B}_l[u](a)$ is *consistent*, i.e. $\mathrm{B}_l[u](a) = \partial u/\partial x_l(a)$ for all $u \in \mathbb{P}^{(2)}$. Due to [2], f is the minimal 2-norm solution of the equations $M(r)f = d$ with

$$M(r) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{x_n^2 y_1 - x_1^2 y_n}{D_1} & \frac{x_1^2 y_2 - x_2^2 y_1}{D_2} & \cdots & \frac{x_{n-1}^2 y_n - x_n^2 y_{n-1}}{D_n} \\ \frac{y_n y_1 (x_n - x_1)}{D_1} & \frac{y_1 y_2 (x_1 - x_2)}{D_2} & \cdots & \frac{y_{n-1} y_n (x_{n-1} - x_n)}{D_n} \\ \frac{y_n y_1 (y_n - y_1)}{D_1} & \frac{y_1 y_2 (y_1 - y_2)}{D_2} & \cdots & \frac{y_{n-1} y_n (y_{n-1} - y_n)}{D_n} \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$[x_i, y_i] = b^i$ and $D_i = D(a, b^{i-1}, b^i)$ for $i = 1, \ldots, n$.

Definition 5 is in agreement with Lemma 2 and with the following statement:

Lemma 3. *The system of equations $M(r)f = d$ related to the ring $r = (b^1, \ldots, b^4)$ around a vertex a is*

a) unsolvable if a is a boundary vertex and

b) solvable if and only if the vertices b^1, a, b^3 as well as b^2, a, b^4 are situated on one straight-line if a is an inner vertex.

We omit the proof of Lemma 3.

Example. For $a = [0, 0]$, we approximate the partial derivative $\partial u/\partial x_1(a) = -0.5403023$ of $u(x_1, x_2) = \sin(1 + 2x_1 + x_2)/(x_2 - 2)$ by $B_1[u](a)$. In Table 1, we use the ring from Fig. 2 a) with $H(a) = 1.3453624/2^i$ for $i = 1, \ldots, 8$.

i	$H(a)$	$B_1[u](a)$	$\partial u/\partial x_1(a) - B_1[u](a)$
1	6.72681 e-1	-0.460947	-7.93549 e-2
2	3.36341 e-1	-0.519906	-2.03960 e-2
3	1.68170 e-1	-0.535183	-5.11974 e-3
4	8.40852 e-2	-0.539023	-1.27939 e-3
5	4.20426 e-2	-0.539983	-3.19584 e-4
6	2.10213 e-2	-0.540222	-7.98508 e-5
7	1.05106 e-2	-0.540282	-1.99563 e-5
8	5.25532 e-3	-0.540297	-4.98822 e-6

Table 1

i	$H(a)$	$B_1[u](a)$	$\partial u/\partial x_1(a) - B_1[u](a)$
1	1.15244	-0.	-0.104569 e-1
2	5.76222 e-1	-0.577975	3.76723 e-2
3	2.88111 e-1	-0.556928	1.66261 e-2
4	1.44055 e-1	-0.545228	4.92589 e-3
5	7.20277 e-2	-0.541620	1.31737 e-3
6	3.60138 e-2	-0.540642	3.39385 e-4
7	1.80069 e-2	-0.540388	8.60568 e-5
8	9.00346 e-3	-0.540324	2.16627 e-5

Table 2

In Table 2, we use the ring from Fig. 2 b) with $H(a) = 2.3048861/2^i$ for $i = 1, \ldots, 8$.

This example indicates the second order of error of the approximations $\mathrm{B}_l[u](a)$ both for the inner and the boundary vertices a, but an analysis of the accuracy of this averaging operator is necessary.

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