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VALUING BARRIER OPTIONS USING THE ADAPTIVE DISCONTINUOUS GALERKIN METHOD

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Abstract

This paper is devoted to barrier options and the main objective is to develop a sufficiently robust, accurate and efficient method for computation of their values driven according to the well-known Black-Scholes equation. The main idea is based on the discontinuous Galerkin method together with a spatial adaptive approach. This combination seems to be a promising technique for the solving of such problems with discontinuous solutions as well as for consequent optimization of the number of degrees of freedom and computational cost. The appended numerical experiment illustrates the potency of the proposed numerical scheme.

1. Introduction

During the last decade, financial models have acquired increasing popularity in option pricing. The valuation of different types of option contracts is very important in modern financial theory and practice – especially exotic options such as discrete barrier options. Most of the analytical formulas for these options is limited by strong assumptions, which led to the application of numerical methods instead.

Therefore, the main goal of this paper is to develop an efficient, robust and accurate numerical method for the barrier option pricing problem, which arises from the concept of the *discontinuous Galerkin* (DG) approach for the space semi-discretization, for more details see [5], and the *backward Euler* scheme for the discretization of the resulting ODE systems. In order to increase the efficiency of the proposed method additionally, this approach is equipped with an *h-adaptivity* technique based on regularity and residual indicators, cf. [1, 2]. The resulting numerical scheme is applied to a standard problem of discrete double barrier option pricing.

2. Barrier option pricing model

In what follows, we consider the double time-independent discrete barrier knock-out option, i.e. option that expires worthless if one of the two barriers has been hit at a monitoring date, see e.g. [1] and [6]. We denote by x the price of an underlying asset (e.g. stock) and by t the time to expiry of the option and let $M := \{0 = t_0^M < t_1^M < \ldots < t_{l-1}^M < t_l^M = T\}$ be the set of monitoring dates and B_- be the lower barrier and B_+ the upper barrier active only at discrete instances $t_l^M \in M$.

Let $\Omega \equiv (0, S_{max})$, $0 < B_- < B_+ < S_{max}$, be a bounded open interval and T stands for maturity. The price $u : Q_T = \Omega \times (0, T) \to I\!R$ of the discrete barrier option satisfies the *Black-Scholes* partial differential equation with initial and boundary conditions:

$$\frac{\partial}{\partial t}u(x,t) - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}u(x,t) - rx\frac{\partial}{\partial x}u(x,t) + ru(x,t) = 0 \quad \text{in } Q_T, \tag{1}$$

$$u(0,t) = 0 \quad \text{and} \quad u(S_{max}, t) = 0, \tag{2}$$

$$u(x,0) = \begin{cases} \max(x-K,0)\cdot\chi_{[B_-,B_+]}, & \text{(call)} \\ \max(K-x,0)\cdot\chi_{[B_-,B_+]}, & \text{(put)} \end{cases}, \quad x \in \Omega, \tag{3}$$

where $\sigma > 0$ and $r > 0$ are constant model parameters denoting the volatility of stock price and the risk-free interest rate, respectively.

From the mathematical point of view the problem (1)–(3) represents a *convection-diffusion-reaction* equation equipped with a set of two homogeneous Dirichlet boundary conditions (2) prescribed at the endpoints of interval $(0, S_{max})$ and with the initial condition (3), where symbol K stands for the strike price and $\chi_{[B_-,B_+]}$ denotes the characteristic function of the barrier interval.

Moreover the discrete monitoring of the contract introduces an updating of the solution $u(x,t)$ at the monitoring dates $t_l^M \in M$:

$$u(x, t_l^M) = \lim_{\varepsilon \to 0+} u(x, t_l^M - \varepsilon)\cdot\chi_{[B_-,B_+]}. \tag{4}$$

3. Discontinuous Galerkin discretization

Let $\mathcal{T}_h$ $(h > 0)$ be a family of *partitions* of the closure $\overline{\Omega} = [0, S_{max}]$ of the domain Ω into N closed mutually disjoint subintervals $I_k = [x_{k-1}, x_k]$ with length $h_k := x_k - x_{k-1}$. Then we set $\mathcal{T}_h = \{I_k,\ 1 \le k \le N\}$ with spatial step $h := \max_{1\le k\le N} h_k$ and call interval I_k an *element*. We additionally assume that the following conditions are satisfied:

$$\exists\, C_q \ge 1 : h_k \le C_q h_{k'} \quad \forall\, I_k, I_{k'} \in \mathcal{T}_h \text{ sharing a node} \quad \text{(local quasi-uniformity)} \tag{5}$$

$$\exists\, k_1, k_2 \in I\!N \text{ such that } x_{k_1} = B_- \text{ and } x_{k_2} = B_+ \quad \text{(barrier consistency)} \tag{6}$$

The DG method can handle different polynomial degrees over elements. Therefore, we assign a positive integer p_k as a *local polynomial degree* to each $I_k \in \mathcal{T}_h$. Then we set the vector $\mathsf{p} = \{p_k, I_k \in \mathcal{T}_h\}$. Over the triangulation $\mathcal{T}_h$ we define the finite dimensional space of discontinuous piecewise polynomial functions:

$$S_{h\mathsf{p}} \equiv S_{h\mathsf{p}}(\Omega, \mathcal{T}_h) = \{v; v|_{I_k} \in P_{p_k}(I_k)\ \forall\, I_k \in \mathcal{T}_h\}, \tag{7}$$

where $P_{p_k}(I_k)$ denotes the space of all polynomials of degree $\le p_k$ on I_k, $I_k \in \mathcal{T}_h$. Consequently, the approximate solution of the continuous problem (1)–(4) is sought in the space $S_{h\mathsf{p}}$.

Let us denote $v(x_k^{\pm}) = \lim_{\varepsilon \to 0+} v(x_k \pm \varepsilon)$. Then we define the *jump* and *average* of v at inner points x_k of Ω by $[v(x_k)] = v(x_k^-) - v(x_k^+)$ and $\langle v(x_k) \rangle = \frac{1}{2}\left(v(x_k^-) + v(x_k^+)\right)$, respectively. We also extend the definition of jump and mean value for endpoints of Ω, i.e. $[v(x_0)] = -v(x_0^+)$, $\langle v(x_0) \rangle = v(x_0^+)$, $[v(x_N)] = v(x_N^-)$ and $\langle v(x_N) \rangle = v(x_N^-)$.

Firstly, we recall the space semi-discrete DG scheme presented in [4] and [5]. To this end we introduce the following bilinear forms:

$$a_h^{\Theta}(u,v) = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \frac{1}{2}\sigma^2 x^2 \frac{\partial u(x,t)}{\partial x} v'(x)\,\mathrm{d}x - \sum_{k=0}^{N} \left\langle \frac{1}{2}\sigma^2 x_k^2 \frac{\partial u(x_k,t)}{\partial x} \right\rangle [v(x_k)] + \Theta \sum_{k=0}^{N} \left\langle \frac{1}{2}\sigma^2 x_k^2\, v'(x_k) \right\rangle [u(x_k,t)], \tag{8}$$

$$b_h(u,v) = -\sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} (\sigma^2 - r)x\, u(x,t)\, v'(x)\,\mathrm{d}x + \sum_{k=0}^{N} H\Big(u(x_k^-,t), u(x_k^+,t)\Big)\, [v(x_k)], \tag{9}$$

$$J_h^{\omega}(u,v) = \sum_{k=0}^{N} \omega_k [u(x_k,t)]\, [v(x_k)]. \tag{10}$$

The crucial item of the DG formulation is the treatment of the linear convection and diffusion terms. For the convection form b_h we treat its terms with the aid of a *numerical flux* H, see [3]. The diffusion form a_h^{Θ} includes *stabilization* terms which are added to the formulation of the problem in order to guarantee the stability of the numerical scheme. According to the value of parameter Θ, we speak of *symmetric* ($\Theta = -1$), *incomplete* ($\Theta = 0$) or *nonsymmetric* ($\Theta = 1$) variants. Furthermore, in order to replace the inter-element discontinuities, the semi-discrete scheme is completed with the *penalty* J_h^{ω} weighted by the penalty parameter function ω_k defined in the spirit of [4]. Let us note that the right-hand side term vanishes due to the prescribed homogeneous Dirichlet boundary conditions in (2).

In order to simplify the notation we define the bilinear form:

$$\mathcal{B}_h^{\Theta}(u,v) := a_h^{\Theta}(u,v) + b_h(u,v) + \alpha J_h^{\omega}(u,v) + (2r - \sigma^2)(u,v), \quad \alpha > 0, \tag{11}$$

where $(\cdot,\cdot)$ denotes inner product and the forms $a_h^{\Theta}(\cdot,\cdot)$, $b_h(\cdot,\cdot)$ and $J_h^{\omega}(\cdot,\cdot)$ are given by (8), (9) and (10), respectively. The value of multiplicative constant α before the penalty form J_h^{ω} depends on the properties of diffusion term, see [4]. Finally, we end up with the following DG formulation for the *semi-discrete solution* $u_h(t) \in S_{h\mathsf{p}}$:

$$\frac{d}{dt}\,(u_h(t), v_h) + \mathcal{B}_h^{\Theta}(u_h(t), v_h) = 0 \quad \forall\, v_h \in S_{h\mathsf{p}}, \forall\, t \in (0,T), \tag{12}$$

which represents an ODE system and due to bilinearity of form (11) we can easily discretize (12) by the implicit Euler method. Let $0 = t_0 < t_1 < \ldots < t_r = T$ be a partition of $[0,T]$ with time steps $\tau_l \equiv t_{l+1} - t_l$, $l = 0, 1, \ldots, r-1$. We define the *approximate solution* of problem (1)–(4) as functions $u_h^l \approx u_h(t_l)$, $t_l \in [0,T]$, $l = 0, \ldots, r-1$, satisfying the following numerical scheme:

$$\left(u_h^{l+1}, v_h\right) + \tau_l \mathcal{B}_h^{\Theta}\left(u_h^{l+1}, v_h\right) = \left(u_h^l, v_h\right) \qquad \forall\, v_h \in S_{h\mathsf{p}}, \tag{13}$$

$$u_h^{l+1} := u_h^{l+1} \cdot \chi_{[B_-,B_+]} \quad \forall\, t_{l+1} \in M, \tag{14}$$

where u_h^0 is $S_{h\mathsf{p}}$-approximation of u^0. The discrete problem (13) is equivalent to a system of linear algebraic equations at each time level $t_{l+1} \in [0, T]$.

4. Mesh adaptation

In this section, we introduce an h-adaptive DG technique for the solution of problem (1)–(4). Since we deal with nonstationary problems, it is suitable to use adaptive mesh refinement during the computation in order to improve the numerical solution and to optimize the number of degrees of freedom and computational cost, consequently.

We start from a uniform coarse grid $\mathcal{T}_{0,h} := \mathcal{T}_h$ and construct at each time instance $t_l \in [0, T]$ a new mesh $\mathcal{T}_{l,h}$ depending on the previous grid $\mathcal{T}_{l-1,h}$ through the following h-adaptation operations: cutting (C) one element I_k into I_{k_1} and I_{k_2} and gluing (G) two elements I_{k_1} and I_{k_2} together into I_k. The described adaptation process has to comply with restrictions on a minimal admissible size of mesh step h_{min}, a maximal admissible size of mesh step h_{max}, a maximal number of elements N_{max} and keeping of local quasi-uniformity (5) and barrier consistency (6), respectively.

The main idea of the proposed h-adaptive strategy is based on

- *mesh refinement* in domains with irregular solution (low regularity) or with high value of residual estimate,
- *mesh coarsening* in domains with solution of high regularity and low value of residual estimate.

The estimation of the regularity of the solution is essential for mesh refinement. The presented approach is based on a measure of inter-element jumps arising from the shock capturing techniques in hyperbolic problems, for a survey see [2].

We have employed the following element-wise *regularity indicator*:

$$g_{I_k}(u_h) := \frac{1}{h_k^{2p_k+1}} \left(\sum_{i=k-1}^{k} [u_h(x_i)]^2 \right), \quad k = 1, \dots N, \tag{15}$$

which recognizes the subdomains of Ω where the solution is smooth ($g_{I_k} \approx 0$) from the areas with discontinuities or with a very steep gradient ($g_{I_k} \gg 1$).

The second key ingredient of the mesh refinement is the *residual estimator* which is chosen proportionally to the strong formulation of the local residue from [1] as

$$r_{I_k}(u_h) := \frac{\partial u_h}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u_h}{\partial x^2} - rx\frac{\partial u_h}{\partial x} + ru_h, \quad I_k \in \mathcal{T}_h. \tag{16}$$

Then the *local* and *global* residual estimators of the approximate solution u_h are defined by $res_{I_k}(u_h) := \|r_{I_k}\|_{L^2(I_k)}$ and $res_G(u_h) := \sqrt{\sum_{I_k \in \mathcal{T}_h} res_{I_k}^2}$, respectively.

Our interest is to find a solution $\widetilde{u_h} \in S_{h\mathsf{p}}$ such that $res_G(\widetilde{u_h}) \leq TOL$, where $TOL > 0$ is a given tolerance. In order to satisfy this condition we prescribe the following stopping criterion for the h-adaptivity: $res_{I_k} \leq \frac{TOL}{N}, \forall\, I_k \in \mathcal{T}_h$, which guarantees the uniform distribution of the global residue.

The whole h-adaptation DG algorithm can be schematically written as

1. let $TOL > 0$, $0 < h_{min} \leq h_{max}$ and N_{max} be given,
2. let $B_-, B_+ \longleftrightarrow \mathcal{T}_{0h}$ and $S_{h\mathsf{p}}$ be set up, let $u^0 \longleftrightarrow u_h^0$ be given,
3. repeat time loop (until $t_l > T$) ($l = 1, \ldots, r$)
   - (a) solve problem (13)–(14) on $\mathcal{T}_{l-1,h} \Longrightarrow u_h^l$,
   - (b) evaluate indicators $g_{I_k}(u_h^l)$, $res_{I_k}(u_h^l)$, $\forall\, I_k \in \mathcal{T}_{l-1,h} \Longrightarrow res_G(u_h^l)$,
   - (c) if $res_G(u_h^l) > TOL \Rightarrow h$-refinement,
     - (C) h-refine elements with $res_{I_k} > \frac{TOL}{N}$,
     - (G) h-derefine elements with $res_{I_k} < \delta\frac{TOL}{N} \wedge g_{I_k}(u_h^l) \approx 0$,
     - (•) construct new mesh $\mathcal{T}_h^{new} \longrightarrow \mathcal{T}_{l-1,h}$ and space $S_{h\mathsf{p}}$, go to (a),
   - (d) if $res_G(u_h^l) \leq \frac{TOL}{\beta} \Rightarrow h$-coarsening,
     - (G) h-derefine elements with $res_{I_k} < \delta\frac{TOL}{N} \wedge g_{I_k}(u_h^l) \approx 0$,
     - (•) construct new mesh $\mathcal{T}_h^{new} \longrightarrow \mathcal{T}_{l-1,h}$ and space $S_{h\mathsf{p}}$, go to (a),

where $\beta > 1$ and $\delta \in (0, 1)$ are user-defined parameters, in our computations they are typically chosen as $\beta = 3.0$ and $\delta = 0.1$.

5. Numerical example

The presented numerical example represents the case of a discrete double barrier call option with the expiration date $T = \frac{8}{12}$ (e.g. 8 months) and the strike price $K = 6.0$. The prescribed barriers are $B_- = 4.0$, $B_+ = 8.0$ and computational domain was set as $\Omega = [0, 9]$. The Black-Scholes model parameters were the risk-free interest rate $r = 1.0y^{-1}$ and volatility $\sigma^2 = 0.01y^{-1}$. The initial uniform mesh with spatial step $h = 0.25$ was adaptively refined according to h-adaptation parameters $h_{min} = 10^{-3}$ and $h_{max} = 0.5$. The time step is $\tau = \frac{1}{120}$ and we consider monthly monitoring. We carried out computations by piecewise quadratic approximations, set $\Theta = 0$ and used the restarted GMRES for the solving of linear systems (13).

Table 1 illustrates the development of the global residue and the number of elements during the computation in comparison with an adapted and uniform mesh. One can easily observe that for approximately the same values of the global residue, it is sufficient to use less elements in the adapted case than for the uniform one. Figure 1 shows the corresponding isolines of option price and global residue in space-time plot with well-resolved monthly monitoring.

6. Conclusion

We have dealt with the numerical solution of the discrete barrier option pricing models, represented by the linear convection-diffusion-reaction equation. We have presented DG approach together with simple h-adaptivity technique. Presented numerical example illustrated the potency of the resulting scheme.

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time (bimonthly)	res_G (adapted)	$\#\mathcal{T}_{lh}^*$	res_G (uniform)	$\#\mathcal{T}_{lh}$
0.000000	19.960869	36	10.888578	120
0.166667	1.498133	178	1.459563	120
0.333333	0.615153	58	0.476494	120
0.500000	0.572154	44	0.475957	120
0.666667	0.119596	58	0.124287	120

Table 1: Comparison of h-adaptive and uniform approach w.r.t. res_G; $\mathcal{T}_{lh}^*$ denotes input meshes after monitoring without the updated h-refinement or h-coarsening.

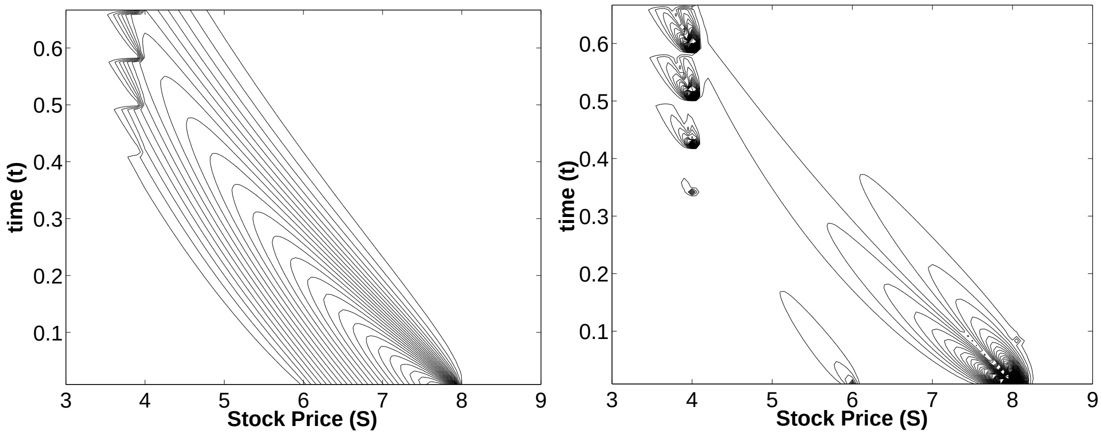

Figure 1: The isolines of price u (left) and corresponding global residue res_G (right).

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