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SHAPE FUNCTIONS AND WAVELETS – TOOLS OF NUMERICAL APPROXIMATION

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Abstract

Solution of a boundary value problem is often realized as the application of the Galerkin method to the weak formulation of given problem. It is possible to generate a trial space by means of splines or by means of functions that are not polynomial and have compact support. We restrict our attention only to RKP shape functions and compactly supported wavelets. Common features and comparison of approximation properties of these functions will be studied in the contribution.

1. Introduction

One of the possibilities to solve boundary value problems is the Galerkin method. Céa's lemma (1964) says that the error in the Galerkin method depends on how well the exact solution is approximated by elements of the trial space. There is a lot of possibilities how to generate such space. For example, it is possible to deal with compactly supported wavelets or with RKP shape functions. The solving of some boundary value problems by using wavelet bases can be found in [5], [2] and by using RKP shape functions for example in [4], [1]. Our aim is to introduce wavelets and RKP shape functions and compare their properties.

The outline of the next text is as follows. Some basic information on the construction and properties of the wavelet basis are presented in Section 2. The construction and properties of the RKP shape functions are described in Section 3. Finally, a comparision of properties of the wavelets and the RKP shape functions is shown in Section 4.

2. Wavelets

Wavelets have grown up not only from theoretical mathematical study but also from practical reasons. The technique of the wavelet transform is used in signal processing. It is a very effective tool, because it gives possibility to change window during the analysing of signal (in contrast with the Fourier transform). It allows to extract information from many different kinds of data, it can help to analyze voice or to compress pictures. It can also serve to analyze variability, to remove noise or to detect significant moments in the time series that are used in economy. In numerical mathematics, the wavelet bases can be used by the solution of boundary values problems, where they provide perfect space and spectral localization. They combine the advantage of the basis used in the FEM with the advantage of the basis used in spectral analysis.

Construction of the wavelet system

A function $\psi \in L^2(R)$ is called the *basic wavelet*, if the condition of stability

$$\int_R \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\,\mathrm{d}\xi < \infty \tag{1}$$

is satisfied.

In this text, we will deal with two types of the basic wavelets – the *scaling function* φ and the *associated wavelet* ψ.

It is possible to receive an orthonormal basis in $L^2(R)$ by means of the *multiresolution analysis* (MRA). The MRA is an efective but not the only one way to obtain an orthogonal wavelet system. Each wavelet that quickly decreases to zero and that is smooth enough can be constructed by it.

In MRA, the spaces $V_j \subset L^2(R)$ $(j \in Z)$ that satisfy

$$\begin{gathered} V_j \subset V_{j+1};\ \bigcap_{j\in Z} V_j = \{0\};\ \bigcup_{j\in Z} V_j = L^2(R); \\ \exists \varphi \in V_0 : \{\varphi(x-k)\}_{k\in Z} \text{ is a complete orthogonal set in } L^2(R); \\ f \in V_0 \Leftrightarrow f(2^j x) \in V_j \end{gathered} \tag{2}$$

are constructed.

It follows from the properties given above that there exists the subspace W_j orthogonal to V_j such that $V_{j+1} = V_j \oplus W_j$. It means that $V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_j$. Next, we put

$$V_j = \{\varphi_{j,k}\}_{j,k\in Z},\ \text{where } \varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k), \tag{3}$$

$$W_j = \{\psi_{j,k}\}_{j,k\in Z},\ \text{where } \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k). \tag{4}$$

If a boundary value problem is solved numerically, it is suitable to generate the trial space by wavelets that have compact support. In this case, the scaling function φ and the associated wavelet have to satisfy

$$\varphi(x) = \sum_{k=0}^{D-1} a_k \varphi_{1,k}(x),\quad \psi(x) = \sum_{k=0}^{D-1} b_k \varphi_{1,k}(x),\ \text{where } b_k = (-1)^k a_{1-k}. \tag{5}$$

Example The class of Daubechies wavelets (including coiflets and symlets) can be received by the MRA. The compactly supported Daubechies wavelet of order 4 together with its scaling function are in Figure 1.

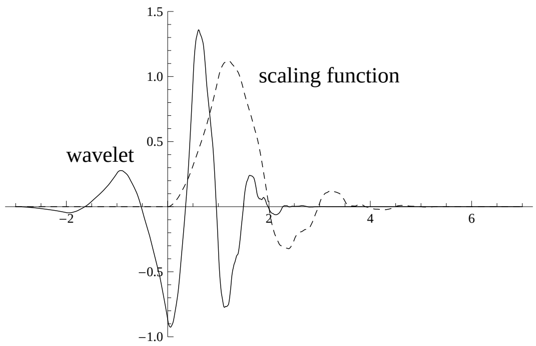

Figure 1: The Daubechies wavelet Db4

Properties of wavelets

1) It holds for wavelets defined by (5) that $\operatorname{supp} \varphi(x) = \langle 0, D-1 \rangle$, $\operatorname{supp} \psi = \langle 1 - \frac{D}{2}, \frac{D}{2} \rangle$. ($D = 2N$ for the Daubechies wavelets of order N.)

2) The functions $\{\varphi_{0,k}\}_{k \in \mathcal{Z}}$, $\{\psi_{l,k}\}_{k \in \mathcal{Z}, l=1,\dots j}$ form an orthonormal basis in $V_{j+1} \subset L^2(R)$. It is possible to express an approximation of a function $u \in L^2(R)$ by means of

$$\tilde{u}(x) = \sum_{k \in \mathcal{Z}} c_{0,k} \varphi_{0,k}(x) + \sum_{l=1}^{j} \sum_{k \in \mathcal{Z}} c_{l,k} \psi_{l,k}(x). \tag{6}$$

3) We can see from the relation (3) that the functions $\{\varphi_{0,k}\}_{k \in Z}$ are translation invariant: $\varphi_{0,k+m}(x) = \varphi_{0,k}(x - m)$.

4) The approximation properties of the MRA are given in the next theorem (see [5]).

Theorem 1. *Let $\{V_j\}$ be the MRA with $\varphi \in L^1(R)$, φ be compactly supported, the value of the Fourier transform $\hat{\varphi}(0) = 1$ and $L \geq 1$, then the next conditions are equivalent*

(a) The Strang-Fix condition of order $L - 1$: Function φ satisfies

$$\frac{\mathrm{d}^q}{\mathrm{d}\xi^q} \hat{\varphi}(2n\pi) = 0, \ n \neq 0, \ n \in Z, \ q = 0, \dots, L-1. \tag{7}$$

(b) The quasi-reproducing condition of order $L - 1$: Function φ satisfies

$$\sum_{k \in Z} k^q \varphi(x - k) = x^q + p_{q-1}(x) \ for \ all \ x \in R, \ q = 0, \dots, L-1. \tag{8}$$

Here p_{q-1} is a polynomial that has order less or equal $q - 1$.

(c) The vanishing moment condition: It holds for the q^{th} moment of the associated wavelet

$$M_q(\psi) = \int_R x^q \psi(x) \, \mathrm{d}x = 0 \quad \forall q = 1, \dots, L-1. \tag{9}$$

(d) There exist coeficients $c_{j,k}$, $j, k \in Z$, and constants C_s, such that it holds for all $u \in W^{L,2}(R)$

$$\|u - \sum_{k \in Z} c_{j,k} \varphi_{j,k}\|_{W^{s,2}(R)} \leq C_s 2^{-j(L-s)} |u|_{W^{L,2}(R)} \ for \ s = 0, \dots, L-1. \tag{10}$$

Remark The construction of orthogonal wavelet bases on the real line was described in the previous text. Note that if boundary value problems are solved, it is necessary to adapt wavelet bases to the interval. Some problems can occur when the wavelets are used directly as trial functions. For example the introduction of Dirichlet boundary conditions is difficult. Lower order wavelets cannot be employed due to the lack of regularity. Also the request for orthogonality in (2) is too strong. It appears better to use Riesz wavelet bases than orthonormal bases given above by solving BVP's. Especially the biorthogonal multiwavelets on the basis of splines are used successfully.

3. RKP-shape functions

Meshless methods were developed to find the solution of boundary value problems for differential equations that describe practical problems such as large deformation, crack propagation or moving boundary problems where it is necessary to overmesh during computation. The fact that meshless methods need no explicitly given mesh avoids or greatly simplifies this meshing task. The trial space is generated by shape functions in meshless methods. There is a lot of meshless methods and each of them constructs the shape functions in a different way. For instance the Reproducing Kernel Particle Method (RKPM) belongs to meshless methods that are based on kernel approximation.

Construction of shape functions

Let $x_1, \dots, x_N$ be particles in $\langle a, b\rangle$, $w(x)$ be a weight function (i.e. continuous, compactly supported function) and $\mathbf{p}(x) = (p_0(x), \dots, p_s(x))$ be a polynomial basis of order s (i.e. components $p_j \in P_{\leq s}$, $s \geq 0$.)

The one dimensional *RKP shape function* $\Phi_j^{[\alpha]}(x)$ of order α, $0 \leq \alpha \leq s$, which is associated with the particle x_j, is defined by

$$\Phi_j^{[\alpha]}(x) = \alpha!\mathbf{p}\left(\frac{x - x_j}{\rho}\right)\mathbf{b}_\alpha^T(x)\, w\left(\frac{x - x_j}{\rho}\right)\Delta x_j. \tag{11}$$

Here $\rho > 0$ is a dilatation parameter, Δx_j is the quadrature weight and vector $\mathbf{b}_\alpha(x)$ is the solution of the linear equations

$$M(x)\mathbf{b}_\alpha^T(x) = \left(\mathbf{p}^{(\alpha)}(0)\right)^T, \tag{12}$$

where $M(x) = \sum_{j=1}^N \mathbf{p}^T\left(\frac{x-x_j}{\rho}\right)\mathbf{p}\left(\frac{x-x_j}{\rho}\right) w\left(\frac{x-x_j}{\rho}\right)\Delta x_j$ and $\mathbf{p}^{(\alpha)}(x) = \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\mathbf{p}(x)$

The vector $\mathbf{b}_\alpha(x)$ is constructed in such a way that the shape functions $\Phi_j^{[\alpha]}(x)$ reproduce polynomials of order $s - \alpha$.

If we use (12), (11) and put $p_\beta\left(\frac{x-x_j}{\rho}\right) = \left(\frac{x-x_j}{\rho}\right)^\beta$, $0 \leq \beta \leq s$, we can see that the condition (12) leads to system

$$\sum_{j=1}^N \left(\frac{x - x_j}{\rho}\right)^\beta \Phi_j^{[\alpha]}(x) = \alpha!\delta_{\beta,\alpha}, \quad 0 \leq \alpha, \beta \leq s. \tag{13}$$

Example The system of reproducing RKP shape functions $\Phi_3^{[0]}$ and $\Phi_3^{[1]}$ is given in Figure 2. They are constructed on the interval $\langle 0, 1\rangle$ for $N = 5$ equidistant particles, $\mathbf{p}(x) = (1, x)$, $w(x) = \begin{cases} (1-x^2)^2 & \text{if } \ |x| \le 1 \\ 0 & \text{if } \ |x| > 1 \end{cases}$ and $\rho = 0.3$.

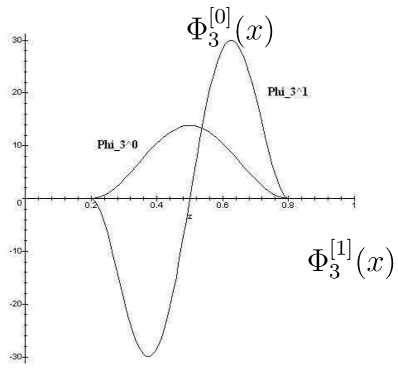

Figure 2: Shape functions

Properties of RKP shape functions
Suppose that RKP shape functions are defined by (11), (12).
1) The continuous version of function $\Phi_0^{[0]}$ satisfies the condition of stability (1) for the basic wavelet (see [3]).
2) The support and smoothness of $\Phi_j^{[0]}$ are the same as the support of the given weight function w.
3) The functions $\Phi_j^{[0]}$ are translation invariant for uniformly distributed particles, i.e. $\Phi_{j+k}^{[0]}(x) = \Phi_j^{[0]}(x - x_k)$, where $x_k = kh, \ k \in Z$ (see [1]).
4) From the conditon (13) one can receive that the shape functions $\Phi_j^{[0]}$ are reproducing of order s i.e. they reproduce polynomials from $P_{\le s}$ exactly (see [3]).
5) It is possible to receive from (13) that $\sum_{j=1}^{N} \Phi_j^{[0]}(x) = 1$ and $\sum_{j=1}^{N} \Phi_j^{[\alpha]}(x) = 0$. It means that the shape functions $\Phi_j^{[\alpha]}, 0 \le \alpha \le s$, form the partition of unity and an approximation of a function $u \in W^{1,2}(\Omega)$ can be supposed in the form

$$\tilde{u}(x) = \sum_{j=1}^{N} c_{0,j}\Phi_j^{[0]}(x) + \sum_{\alpha=1}^{s}\sum_{j=1}^{N} c_{\alpha,j}\Phi_j^{[\alpha]}(x). \qquad (14)$$

6) Because the property "reproducing order" is a particular case of "quasi-reproducing order", the error of approximation can be determined from the Strang-Fix theorem (see [1]).

Theorem 2. *Let particles $\{x_i\}$ be uniformly distributed, $\Phi_j^{[0]} \in W^{q,2}(R)$, $q \ge 0$, be reproducing of order s. Then for each $u \in W^{k+1,2}(R)$ there are $C, c_j \in R$ such that*

$$\|u - \sum_{j\in Z} c_j\Phi_j^{[0]}\|_{W^{s,2}} \le C\, h^{k+1-s}\|u\|_{W^{k+1,2}} \ for \ 0 \le s \le \min\{q, k+1\}. \qquad (15)$$

4. Conclusion

In this contribution the construction of compactly supported wavelet and RKP shape function systems is described. Then a short overview of properties of these systems is given. It is possible to say that even though these systems are built in different ways, they have some common features.

For example: The basic functions $\Phi_0^{[0]}$ behave similarly as the scaling functions φ. It is possible to obtain the constructed systems from these basic functions using translation and dilatation. The basic functions are able to approximate polynomials of the order, which corresponds to the order of reproducing conditions that they satisfy. The functions $\psi_{j,k}$ and $\Phi_j^{[\alpha]}, \alpha \neq 0$, satisfy the vanishing moment condition. It is possible to carry out the estimate of approximation errors using the Strang-Fix theorem.

However, it is possible to find some differences between wavelet bases and RKP shape functions that are used for solution of BVP's. For example, biorhotgonal wavelet bases are Riesz bases, but the sequence $\{\Phi_I^{[\alpha]}(x), \alpha \geq 0\}$ is only a frame. Wavelet basis provides the possibility to compute effectively coefficients of a stiffness matrix, but the RKP shape functions do not offer any similar advantage. On the other hand, it is possible to construct RKP shape functions that have the desired order of continuity and that are not linked to any explicitly given mesh.

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