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OPTIMIZATION OF PLOUNGER CAVITY

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Abstract

In the contribution we present a problem of shape optimization of the cooling cavity
of a plunger that is used in the forming process in the glass industry. A rotationally
symmetric system of the mould, the glass piece, the plunger and the plunger cavity is
considered. The state problem is given as a stationary heat conduction process. The
system includes a heat source representing the glass piece that is cooled from inside
by water flowing through the plunger cavity and from outside by the environment
surrounding the mould. The design variable is the shape of the inner surface of the
plunger cavity.

The cost functional is defined as the squared $L^2$ norm of the difference between
a prescribed constant and the temperature on the outward boundary of the plunger.

1. Introduction

This work deals with the optimal design of the shape of a plunger cavity that
controls the cooling of a glass piece during the manufacturing process. The aim of
the optimization is to find such a shape of the inner plunger cavity that allows for
cooling in such a way that a constant distribution of the temperature is achieved
across the surface of the moulding device at the moment of separation of the plunger
from the moulded piece.

2. Formulation of the problem

We rotate the system to the horizontal position to be able to describe the optim-
ized plunger cavity surface by a function of one variable.

We define

$$F^*_2(x) = \begin{cases} 0 & \text{for } x \in [0, x^*_2) \\ f^*_2(x) & \text{for } x \in [x^*_2, 1) \end{cases},$$

where $x^*_2 \in [s_{\text{min}}, 1]$ ($s_{\text{min}} > 0$ is a fixed constant given by the minimal thickness
of the plunger wall), $f^*_2 \in C((0, 1], [x^*_2, 1])$, $f^*_2(x^*_2) = 0$ and $0 \leq f^*_2(x) \leq f_1(x) - s_{\text{min}}$, $|f^*_2'(x)| < C_D$ for $x \in [x^*_2, 1]$, where $f_1$ is a fixed function. Further we assume that

$$a \leq f^*_2(x) - s_2$$

for $x \in [x^*_2, 1]$, where $a > 0$ represents the radius of a supply tube and
\[ s_2 > 0 \] is the minimal admissible split width between the inner wall of the plunger cavity and the water supply tube, and \( x_3^e \in [x_2, 1] \) is the deepness of the insertion of the tube.

Further we define the set of admissible functions as

\[
U_{ad}^e = \{ F_2^e(x) \in C^{(0),1}([0, 1]) ; F_2^e(x) = \begin{cases} 0 & \text{for } x \in [0, x_2^e] \\ f_2^e(x) & \text{for } x \in [x_2^e, 1] \end{cases} ,
\]

\[
x_2^e \in [s_{\text{min}}, 1], \ s_{\text{min}} > 0, \ f_2^e \in C^{(0),1}([x_2^e, 1]), \ f_2^e(x_2^e) = 0,
0 \leq f_2^e(x) \leq f_1(x) - s_{\text{min}}, \ |f_2^e'(x)| < C_D \text{ for } x \in [x_2^e, 1],
\]

\[
f_1 \text{ given, } a \leq f_2^e(x) - s_2 \text{ for } x \in [x_3^e, 1], \ a > 0, \ s_2 > 0, \ x_3^e \in [x_2, 1] \},
\]

where the function \( F_2^e \) describes the technological constraint for the inner cavity surface.

We assume the region \( \Omega_{Pl}^e \) that depends on the design function \( F_2^e(x) \), and that is defined by the formula

\[
\Omega_{Pl}^e = \{ (x, r) \in R^2 ; F_2^e(x) < r < f_1(x), \ \text{for } x \in [0, 1] \}.
\]

Denote by \( \Theta \) the set of all admissible regions \( \Omega_{Pl}^e \subset R^2 \), i.e., regions characterized by \( F_2^e \in U_{ad}^e \). Let us define the convergence on the set \( \Theta \). Since each \( \Omega_{Pl}^e \) is uniquely related to \( F_2^e \), we can say that a sequence \( \Omega_{Pl}^e \in \Theta \) converges to a region \( \Omega_{Pl}^e \in \Theta \) if and only if the sequence of functions \( F_2^e(x) \) converges uniformly in \([0, 1]\) to the function \( F_2^e(x) \) that defines \( \Omega_{Pl}^e \).

Let us consider the union of four planar regions \( \Omega = \Omega_{Mo} \cup \Omega_{Gl} \cup \Omega_{Pl}^e \cup \Omega_{Ca}^e \) that represents the planar cross section of the mould, the glass piece, the plunger and the cooling channel of the plunger (see Figure 2).

Furthermore, we denote by \( \Gamma_1 \) the boundary between the plunger \( \Omega_{Pl}^e \) and the moulded piece \( \Omega_{Gl} \) and \( \Gamma_2^e \) the boundary between the plunger \( \Omega_{Pl}^e \) and the plunger cavity \( \Omega_{Ca}^e \). We denote by \( \Gamma_3 \) the part of the boundary connecting the mould, the moulded piece and the plunger with the presser, by \( \Gamma_4 \) a part of the axis of symmetry (see Figure 2), by \( \Gamma_5 \) the part of the boundary formed by the tube. \( \Gamma_6 \) is the notation for the part of the boundary between the moulded piece \( \Omega_{Gl} \) and the mould \( \Omega_{Mo} \).
$\Gamma_7$ is the outward boundary of the mould, which is surrounded by an external environment. $\Gamma_{in}$ denotes the part of the boundary, where the cooling water comes into the cooling channel of the plunger, and $\Gamma_{out}$ stands for the part of the boundary, where the water exits the channel.

In the three dimensional region $\Omega_{Ga}^e$, which is created by the rotation of $\Omega_{Ga}^e$ around the $x$ axis, we assume an incompressible potential water flow that is rotationally symmetric with respect to the $x$ axis. We split the boundary $\partial \Omega_{Ga}^e$ into the union of four parts as

$$\partial \Omega_{Ga}^e = \Gamma_{2}^{3D} \cup \Gamma_{5}^{3D} \cup \Gamma_{in}^{3D} \cup \Gamma_{out}^{3D},$$

where $\Gamma_{2}^{3D}$, $\Gamma_{5}^{3D}$, $\Gamma_{in}^{3D}$, and $\Gamma_{out}^{3D}$ denote the respective parts of the boundary of $\partial \Omega_{Ga}^e$ created by the rotation of $\Gamma_2$, $\Gamma_5$, $\Gamma_{in}$, and $\Gamma_{out}$ around the $x$ axis.

The potential $\Phi$ describing the water flow is given as a solution of the Neumann problem

$$\Delta \Phi = 0 \quad \text{in} \quad \Omega_{Ga}^e, \quad (3)$$

$$\frac{\partial \Phi}{\partial n} = g \quad \text{on} \quad \partial \Omega_{Ga}^e, \quad (4)$$

where $g \in L^2(\partial \Omega_{Ga}^e)$, representing the normal component of the water flow velocity at the entrance to and the exit from the plunger cavity, is in the form

$$g = \begin{cases} 
0 & \text{on} \ \Gamma_{2}^{3D} \cup \Gamma_{5}^{3D}, \\
\dot{h}_{velo}^{in} & \text{on} \ \Gamma_{in}^{3D}, \\
\dot{h}_{velo}^{out} & \text{on} \ \Gamma_{out}^{3D},
\end{cases} \quad (5)$$

$h_{velo}^{in}$ is the normal velocity at the entrance $\Gamma_{in}^{3D}$ ($\dot{h}_{velo}^{in} < 0$) and $h_{velo}^{out}$ is the normal velocity at the exit $\Gamma_{out}^{3D}$. Further we assume

$$\int_{\Gamma_{in}^{3D} \cup \Gamma_{out}^{3D}} g \, dS = 0. \quad (6)$$
The variational formulation for the potential function has the form:

We look for the function \( \Phi \in H^1(G_{Ca}^e) \) such that

\[
\int_{G_{Ca}^e} \left( \frac{\partial \Phi}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{\partial \Phi}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + \frac{\partial \Phi}{\partial x_3} \frac{\partial \varphi}{\partial x_3} \right) dV = \int_{\Gamma_{in}^3 \cup \Gamma_{out}^3} g \varphi \, dS \quad \forall \varphi \in H^1(G_{Ca}^e). \tag{7}
\]

In the cavity \( G_{Ca}^e \), the flowing water velocity field \( \mathbf{u} = (u_1, u_2, u_3) \) is given as

\[
\mathbf{u} = \text{grad} \, \Phi . \tag{8}
\]

**Theorem 1. (existence and uniqueness of the velocity field)** Under the assumption (6) there exists a unique velocity field of the form (8) satisfying

\[
\| \mathbf{u} \|_{L^2(G_{Ca}^e)} \leq c \left( \| h_{velo}^{in} \|_{L^2(\Gamma_{in}^3)} + \| h_{velo}^{out} \|_{L^2(\Gamma_{out}^3)} \right), \tag{9}
\]

where

\[
\| \mathbf{u} \|_{L^2(G_{Ca}^e)} = \left\| \sqrt{u_1^2 + u_2^2 + u_3^2} \right\|_{L^2(G_{Ca}^e)}. \tag{10}
\]

**Proof.** See [3]. \( \square \)

Let us consider the union of four regions \( G = G_{Mo} \cup G_{Gl} \cup G_{Pli} \cup G_{Ca}^e \) that is created by the rotation of the union \( \Omega = \Omega_{Mo} \cup \Omega_{Gl} \cup \Omega_{Pli} \cup \Omega_{Ca}^e \) around the \( x \) axis.

We split \( \vartheta \), the searched function representing the distribution of the temperature, into four functions

\[
\vartheta = \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 , \tag{11}
\]

where

\[
\vartheta_i = \begin{cases} 
\vartheta |_{G_i} & \text{in } G_i \\
0 & \text{in } G \setminus G_i 
\end{cases} \quad \text{for } i = 0, 1, 2, 3 , \tag{12}
\]

\((G_0 \equiv G_{Pli}, G_1 \equiv G_{Gl}, G_2 \equiv G_{Ca}^e, G_3 \equiv G_{Mo}).\)

Further we denote by \( \vartheta_{ij}^{3D} \) the trace of the solution \( \vartheta_i \) on the boundary \( \Gamma_{ij}^{3D} \) if \( \Gamma_{ij}^{3D} \) is a part of the boundary of \( G_i \) for \( i = 0, 1, 2, 3 \), \( j = 1, 2, 3, 4, 5, 6, 7, 8, 9 \) \((\Gamma_{8}^{3D} = \Gamma_{in}^{3D}, \Gamma_{9}^{3D} = \Gamma_{out}^{3D}).\)

By virtue of the rotational symmetry of both the state problem and the function \( \vartheta \), the state problem can be formulated variationally in two dimensions. We define the operators

\[
\text{Energy}_{velo}^\Omega(\vartheta, \mathbf{w}, \psi) = c_v \vartheta_2 \int_{G_{Ca}^e} \left( \frac{\partial \vartheta_2}{\partial x} \mathbf{w}_1 + \frac{\partial \vartheta_2}{\partial r} \mathbf{w}_2 \right) \psi \, r \, d\Omega , \tag{13}
\]

\[
\text{Energy}_{cond}^\Omega(\vartheta, \psi) = k_0 \int_{G_{Gl}} \left( \frac{\partial \vartheta_0}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_0}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega +
\]

\[
+ \quad k_1 \int_{G_{Pli}} \left( \frac{\partial \vartheta_1}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_1}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega , \tag{14}
\]

177
\[ + k_2 \int_{\Omega_{Ca}} \left( \frac{\partial \vartheta_2}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_2}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega + \]
\[ + k_3 \int_{\Omega_{Mo}} \left( \frac{\partial \vartheta_3}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \vartheta_3}{\partial r} \frac{\partial \psi}{\partial r} \right) r \, d\Omega , \]

Environment_{\Omega}(\vartheta, \psi) = \int_{\Gamma_7} \alpha \vartheta_4 |r \vartheta_3| r \, d\Gamma , \quad (15)

Source_{\Omega}(\psi) = g_1 \int_{\Omega_{Gl}} q \vartheta_4 r \, d\Omega , \quad (16)

Coeff_{\Omega}(\psi) = \int_{\Gamma_1} \beta_1 \psi r \, d\Gamma + \int_{\Gamma_6} \beta_6 \psi r \, d\Gamma + \int_{\Gamma_7} \alpha \vartheta_3 \psi r \, d\Gamma , \quad (17)

where \( c_v \) is the specific heat capacity per unit volume, \( g_1 \) is the density of glass, \( g_2 \) is the density of water, \( w_1, w_2 \) are the water velocity field components expressed in cylindrical coordinates, \( k_0, k_1, k_2, k_3 \) are the coefficients of thermal conductivity, \( \alpha \) is the coefficient of heat-transfer between the mould and the environment, \( \vartheta_4 \) is the temperature of the environment, \( \beta_1, \beta_6 \) are the average power conversion of the unit volume of the glass body (see [4, page 128]) and \( q \) is the density of heat sources. Further we denote by

\[
A_{\Omega}(\vartheta, w, \psi) = \text{Energy}^{velo}_{\Omega}(\vartheta, w, \psi) + \text{Energy}^{cond}_{\Omega}(\vartheta, \psi) + \text{Environment}_{\Omega}(\vartheta, \psi) \tag{18}
\]

and

\[
F_{\Omega}(\psi) = \text{Source}_{\Omega}(\psi) + \text{Coeff}_{\Omega}(\psi) . \tag{19}
\]

We introduce the weighted Sobolev space \( H^1_i(\Omega_i) \) (see [2]) provided with the norm

\[
\|v\|_{1,r,\Omega_i} = \left( \int_{\Omega_i} \left[ \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + v^2 \right] r \, d\Omega \right)^{\frac{1}{2}} \quad i = 0, 1, 2, 3 , \tag{20}
\]

\((\Omega_0 \equiv \Omega_{Fl}, \Omega_1 \equiv \Omega_{Gl}, \Omega_2 \equiv \Omega_{Ca}, \Omega_3 \equiv \Omega_{Ma}).\)

Further we introduce

\[
H(\Omega) = \{ \vartheta; \vartheta \text{ defined in (12), } \vartheta_i \in H^1_i(\Omega_i) \text{ for any } i = 0, 1, 2, 3, \vartheta_3|\Gamma_6 = \vartheta_1|\Gamma_6, \vartheta_1|\Gamma_1 = \vartheta_0|\Gamma_1, \vartheta_0|\Gamma_2 = \vartheta_2|\Gamma_2 \} ,
\]

where \( \vartheta_i|\Gamma_j \) denotes the trace of the function \( \vartheta_i \) on the boundary \( \Gamma_j \).

We define the norm in \( H(\Omega) \) as

\[
\|\vartheta\|_H = \left( \|\vartheta_0\|_{1,r,\Omega_0}^2 + \|\vartheta_1\|_{1,r,\Omega_1}^2 + \|\vartheta_2\|_{1,r,\Omega_2}^2 + \|\vartheta_3\|_{1,r,\Omega_3}^2 \right)^{\frac{1}{2}} . \tag{21}
\]

**Theorem 2.** The set \( H(\Omega) \) with the norm (21) is a Hilbert space.
We denote by $H^*(\Omega)$ the dual space to the space $H(\Omega)$ with the norm
\[ \|\psi\|_{H^*} = \sup_{\varphi \neq 0} \frac{A_\Omega(\varphi, w, \psi)}{\|\varphi\|_H}. \]
We define the sets
\[ \Omega_H = \Omega \cup \Gamma_3 \cup \Gamma_{in} \cup \Gamma_{out} \]
and
\[ \mathcal{V}^2D = \{ v \in C^\infty(\Omega_H); v|_{\Gamma_3 \cup \Gamma_{in} \cup \Gamma_{out}} = 0 \}. \]

Let $H_0(\Omega)$ be the closure of the set $\mathcal{V}^2D$ in $H(\Omega)$.

We assume the existence of a function $\vartheta \in H(\Omega)$ such that
\[ \vartheta|_{\Gamma_{in}} = 288 \quad \text{on } \Gamma_{in}, \quad (22) \]
\[ \vartheta|_{\Gamma_{out}} = h_{out} \quad \text{on } \Gamma_{out}, \quad (23) \]
\[ \vartheta|_{\Gamma_3} = h_3 \quad \text{on } \Gamma_3, \quad (24) \]
where $h_3 \in C(\Gamma_3)$ is a given function representing the steady temperature on the boundary $\Gamma_3$ (see Figure 2) and $h_{out} \in C(\Gamma_{out})$ is a given function representing the temperature distribution on the cavity output $\Gamma_{out}$.

We use the variational formulation of the energy equation to formulate

**The State Problem:**
We look for the function $\vartheta \equiv \vartheta(F^e_2) \in H(\Omega)$ such that
\[ A_\Omega(\vartheta, w^e, \psi) = F_\Omega(\psi) \quad \forall \psi \in H_0(\Omega), \quad (25) \]
\[ \vartheta - \vartheta^c \in H_0(\Omega), \quad (26) \]
where $F^e_2 \in U_{ad}$ and $w^e$ is the corresponding flow pattern given as the gradient of the solution to (7).

**Remark.** The state problem is solved in two steps. First, the potential $\Phi$ of the water velocity is found as a solution of the problem (7) in the region $G^e_{Ca}$. The components of the velocity field $u$ are computed from (8), transformed to cylindrical coordinates and substituted into (13). Then the distribution of the temperature $\vartheta$ in the whole system $\Omega$ is found as the solution of the state problem (25), (26).

**Theorem 3.** (the existence and uniqueness of the solution of the state problem)
The state problem (25), (26) has a unique solution $\vartheta(F^e_2)$ for each $F^e_2 \in U^e_{ad}$ and the associated flow pattern $w^e$ obtained as the gradient of the unique solution of (7), moreover, there exists a constant $C > 0$ such that
\[ \|\vartheta(F^e_2)\|_H \leq C\|F_\Omega\|_{H^*}. \quad (27) \]

**Proof.** It is sufficient to verify the assumptions of the Lax-Milgram Theorem (see [3]).
We formulate the **problem of the optimal design for the plunger cavity shape**: We define the **cost functional** as

\[
J^S(F_2^e) = \|\vartheta(F_2^e)|_{\Gamma_1} - T_{\Gamma_1}\|^2_{0,r,\Gamma_1},
\]

(28)

where \(\vartheta(F_2^e)|_{\Gamma_1}\) is the \(\Gamma_1\)-trace of the solution \(\vartheta(F_2^e)\) of the state problem (25), (26) in the region \(\Omega_{F_2}\), where \(T_{\Gamma_1}\) is a given constant representing the known optimal temperature of the plunger surface. We look for the **optimal design** \(F_{Opt} \in U^{e}_{ad}\) such that

\[
J^S(F_{Opt}) \leq J^S(F_2^e) \quad \forall F_2^e \in U^{e}_{ad}.
\]

(29)

**Theorem 4.** The optimal design problem (29) has at least one solution.

*Proof.* We refer to Theorem 2.1 [1, page 29], see [3].

**Remark.** A sensitivity analysis can be performed on the basis of temperature evaluation along the boundary \(\Gamma_1\). Let us introduce a homeomorphism between the outward plunger boundary \(\Gamma_1\) and the plunger cavity boundary \(\Gamma_2^e\) defined by the gradient lines of the temperature field in the plunger. In the parts of \(\Gamma_1\) where we need to decrease the temperature, we narrow “the wall” by moving the points of \(\Gamma_2^e\) along the gradient lines to locally achieve more intensive cooling. On the other hand, in places of \(\Gamma_1\) where we need higher temperature, we increase “the wall thickness” to locally decrease the intensity of cooling. By the term “the wall thickness” we understand the length of the temperature gradient line that connects the related points of \(\Gamma_1\) and \(\Gamma_2^e\).

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**References**


