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A PRIORI DIFFUSION-UNIFORM ERROR ESTIMATES FOR SINGULARLY PERTURBED PROBLEMS: MIDPOINT-DG DISCRETIZATION

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Abstract

We deal with a nonstationary semilinear singularly perturbed convection–diffusion problem. We discretize this problem by discontinuous Galerkin method in space and by midpoint rule in time. We present diffusion–uniform error estimates with sketches of proofs.

1. Introduction

Our aim is development of sufficiently robust, accurate and efficient numerical schemes for solving nonlinear singularly perturbed convection–diffusion equations, which describe many important topics, e.g. fluid dynamics.

Singularly perturbed convection–diffusion equations represent very difficult problems, since these problems lie on the edge between elliptic and hyperbolic problems. From numerical point of view these problems are unpleasant, since they have steep gradients or discontinuities in the solution even for smooth data. To overcome these difficulties we employ discontinuous Galerkin method, which uses piecewise discontinuous polynomial functions. It seems that such a weaker inter–element connection partially suppresses spurious oscillations in the discrete solution, which are present in the standard finite element solution.

Applying standard parabolic techniques to this problem we obtain diffusion dependent error estimates – typically with the constant $e^{1/\varepsilon}$, where ε is the diffusion parameter, see e.g. [1] or [4]. In practical cases from compressible fluid dynamics, where ε is about 10^{-5} to 10^{-9}, these error estimates are useless.

Our aim is to derive a priori error estimates that are uniform with respect to the diffusion parameter. A majority of analysis of singularly perturbed problems devoted to the uniform a priori error estimates concerns linear problems only, see e.g. [5].

The technique, how to overcome the nonlinearity in the convective part, is presented in [8], with applications to the explicit time stepping schemes. The technique is based on the linearization of the problem by Taylor expansion, where the problem is divided into linear part and higher order nonlinear reminder. How to deal with the linear part is known from the analysis of purely linear problems. The analysis of the nonlinear reminder is more tricky and takes advantage of higher order of the reminder and of higher order of the error at previous time levels. In [6] we can find the extension of this result to the semidiscrete problem and to the backward Euler method, where (in contrast to explicit schemes) one needs higher order of the error at the actual time level and not at the previous one. This problem is solved by continuous mathematical induction. This paper extends the technique from [6] to midpoint rule.

2. Continuous problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain and $T > 0$. We set $Q_T = \Omega \times (0,T)$. Let us consider the following problem: Find $u : Q_T \to \mathbb{R}$ such that

$$\begin{aligned} &\frac{\partial u}{\partial t} + \nabla \cdot f(u) - \varepsilon \, \Delta u = g \quad \text{in } Q_T, \qquad (1)\\ &u\,\big|_{\partial\Omega\times(0,T)} = 0,\\ &u(x,0) = u^0(x), \quad x \in \Omega. \end{aligned}$$

We assume $f = (f_1, \dots, f_d)$, $f_s \in C^2(\mathbb{R})$, $f_s(0) = 0$, $s = 1, \dots, d$ represents convective terms, $\varepsilon \geq 0$, $g \in C([0,T]; L^2(\Omega))$ and $u^0 \in L^2(\Omega)$ is an initial condition. We assume that the weak solution of (1) is sufficiently regular, namely,

$$u \in W^{1,\infty}(0,T; H^{p+1}(\Omega)) \cap W^{2,\infty}(0,T; H^2(\Omega)), \quad u^{(3)} \in L^\infty(0,T; L^2(\Omega)), \qquad (2)$$

where $u^{(k)} = \partial^k u / \partial t^k$, an integer $p \geq 1$ will denote a given degree of polynomial approximations in space.

3. Discrete problem

To simplify the expressions we use the notation $(\cdot,\cdot)$ for L^2 scalar product and $\|\cdot\|$ for L^2 norm. We employ the *symmetric interior penalty Galerkin* (SIPG) method for the space semi-discretization of (1), for details see [2]. Let $\mathcal{T}_h$ $(h > 0)$ be a partition of $\overline{\Omega}$ into a finite number of closed d-dimensional simplices K with mutually disjoint interiors. Let $S_h = \{w; w|_K \in P_p(K) \ \forall K \in \mathcal{T}_h\}$ denote the space of discontinuous piecewise polynomial functions of degree p on each $K \in \mathcal{T}_h$. Then we say that the function $u_h \in C^1(0,T; S_h)$ is the *semi-discrete approximate solution* of (1) if it satisfies the conditions

$$\left(\frac{\partial u_h}{\partial t}(t), w\right) + \varepsilon A_h(u_h(t), w) + b_h(u_h(t), w) = \ell_h(w)\,(t) \quad \forall\, w \in S_h,\ \forall\, t \in [0,T], \qquad (3)$$

and $(u_h(0), w) = (u^0, w)\ \forall\, w \in S_h$. The bilinear form A_h represents the diffusion term with a sufficiently large interior and boundary penalty, b_h is a nonlinear form representing convective term based on the numerical fluxes well known from the finite volume method and ℓ_h represents the source term. For the exact definition of forms A_h, b_h and ℓ_h see e.g. [2]. We assume the numerical fluxes H to be Lipschitz continuous, conservative and consistent. Moreover, we assume that the numerical fluxes are E–fluxes:

$$(H(v, w, n) - f(q) \cdot n)(v - w) \geq 0, \quad \forall v, w \in \mathbb{R},\ \forall q \text{ between } v \text{ and } w, \tag{4}$$

where $n \in \mathbb{R}^d$ is an unit vector.

We find that the weak solution of (1) with property (2) satisfies the identity

$$\left(\frac{\partial u}{\partial t}(t), w\right) + \varepsilon A_h(u(t), w) + b_h(u(t), w) = \ell_h(w)\,(t) \tag{5}$$

for all $w \in S_h$ and all $t \in (0, T)$.

For simplicity we assume time partition $t_m = m\tau$, $m = 0, \dots, r$ with the time step $\tau = T/r$. To simplify the future expressions we set the notation $v^m = v(t_m)$.

Definition 1. *We say that the set of functions $U^m \in S_h$, $m = 0, \dots, r$ is an approximate solution of problem (1) obtained by midpoint–DGFE scheme if*

$$\begin{aligned}(U^m - U^{m-1}, w) + \frac{\tau\varepsilon}{2} A_h(U^m + U^{m-1}, w) + \tau b_h\left(\frac{U^m + U^{m-1}}{2}, w\right) \qquad \qquad (6)\\ = \tau \ell_h(w)(t_{m-1} + \tau/2) \quad \forall w \in S_h,\\ (U^0, w) = (u^0, w) \quad \forall w \in S_h.\end{aligned}$$

4. Error estimates

We denote the energy norm $|||w|||^2 := A_h(w, w)\ \forall w \in S_h$. Note that the inverse inequality takes the following form $|||w||| \leq Ch^{-1}\|w\|$ for $w \in S_h$. Let Π be the L^2 orthogonal projection on S_h.

We summarize the properties of the forms A_h and b_h.

Lemma 1. *Let u satisfy (2). Then*

$$A_h(v, w) \leq C|||v|||\,|||w||| \quad \forall v, w \in S_h, \tag{7}$$

$$A_h(u(t_{m-1} + s/2), w) - A_h\left(\frac{u(s) + u^{m-1}}{2}, w\right) \leq C\tau^2 |||w||| \quad \forall w \in S_h,\ \forall s \in [t_{m-1}, t_m], \tag{8}$$

$$A_h(\Pi u - u, w) \leq Ch^p |||w||| \quad \forall w \in S_h. \tag{9}$$

The proof of (7) and (9) can be done in a similar way as in [3, Lemma 9]. The proof of (8) can be done similarly as in [7, Lemma 4.3].

Lemma 2. *Let u satisfy (2). Then*

$$b_h(v,w) - b_h(\bar{v},w) \le C\|v-\bar{v}\| \, \|\!|w|\!\| \quad \forall v,\bar{v},w \in S_h \tag{10}$$

$$b_h(u(t_{m-1}+s/2),w) - b_h\left(\frac{u(s)+u^{m-1}}{2},w\right) \le C\tau^2 \|\!|w|\!\| \quad \forall w \in S_h, \tag{11}$$

$$\forall s \in [t_{m-1},t_m],$$

$$b_h(v,v-\Pi u) - b_h(u,v-\Pi u) \le C\left(1+\frac{\|v-u\|_\infty^2}{h^2}\right)(h^{2p+1}+\|v-\Pi u\|^2) \tag{12}$$

$$\forall v \in S_h.$$

The proof of (10) can be found in [3], the proof of estimate (11) uses the regularity of arguments with respect to space and standard error estimates and (12) can be found in [6].

Our goal is to investigate the error estimates of the approximate solution U^m, $m = 0,\dots,r$ obtained by the method (6). To do this we employ the strategy of continuous extension of the discrete solution mimicking to the strategy in [6].

Definition 2. *Let $s \in (0,\tau]$. We say that the function $U(t_{m-1}+s) \in S_h$ is a continuated approximate solution of problem (1) obtained by midpoint–DGFE scheme if*

$$\begin{aligned}(U(t_{m-1}+s) - U^{m-1},w) &+ \frac{s\varepsilon}{2}A_h(U(t_{m-1}+s)+U^{m-1},w)\\ &+ sb_h\left(\frac{U(t_{m-1}+s)+U^{m-1}}{2},w\right) = s\ell_h(w)(t_{m-1}+s/2) \quad \forall w \in S_h.\end{aligned} \tag{13}$$

It is obvious that $U(t_m) = U^m$.

We denote the left–hand side and right–hand side from Definition 2

$$B_s^m(v,w) = (v-U^{m-1},w) + \frac{s\varepsilon}{2}A_h(v+U^{m-1},w) + sb_h\left(\frac{v+U^{m-1}}{2},w\right), \tag{14}$$

$$L_s^m(w) = s\ell_h(w)(t_{m-1}+s/2). \tag{15}$$

We shall show that B_s^m is strongly monotone on S_h:

$$\begin{aligned}B_s^m(v,v-w) - B_s^m(w,v-w) &\ge \|v-w\|^2 + \frac{s\varepsilon}{2}\|\!|v-w|\!\|^2 - Cs\|v-w\|\,\|\!|v-w|\!\|\\ &\ge \left(1+\frac{s\varepsilon}{h^2}-\frac{Cs}{h}\right)\|v-w\|^2 = M\|v-w\|^2\end{aligned} \tag{16}$$

for sufficiently small s respectively τ. We shall show that B_s^m is Lipschitz continuous on S_h:

$$\begin{aligned}B_s^m(v,w) - B_s^m(\bar{v},w) &\le \|v-w\|\,\|w\| + C\frac{s\varepsilon}{2}\|\!|v-\bar{v}|\!\|\,\|\!|w|\!\| + Cs\|v-\bar{v}\|\,\|\!|w|\!\| \\ &\le \left(1+\frac{Cs\varepsilon}{h^2}+\frac{Cs}{h}\right)\|v-\bar{v}\|\,\|w\| = C\|v-\bar{v}\|\,\|w\|.\end{aligned} \tag{17}$$

Since right–hand side L_s^m is evidently Lipschitz continuous, we can employ nonlinear Lax–Milgram lemma to prove the existence of the continuated discrete solution and classical discrete solution, respectively.

Now we should show that the continuated discrete solution is really continuous. Since the proof is the same at each time interval $(t_{m-1}, t_m]$, we show it for the simplicity only on the first one. Let $t, s \in (0, \tau]$. Then

$$\begin{aligned} M\|U(t) - U(s)\|^2 &\leq B_t^1(U(t), U(t) - U(s)) - B_t^1(U(s), U(t) - U(s)) \\ &= L_t^1(U(t) - U(s)) - L_s^1(U(t) - U(s)) \\ &\quad + B_s^1(U(s), U(t) - U(s)) - B_t^1(U(s), U(t) - U(s)). \end{aligned} \tag{18}$$

Since the terms on the second and third row tend to zero if $|t - s|$ tends zero we obtain continuity. Analogically we can prove the continuity at 0+. Since the exact solution u is continuous and since we have continuity on the closed interval $[0, T]$, we can see that the error $U(t) - u(t)$ is uniformly continuous.

As the final step we shall derive the error estimate of the continuated solution at arbitrary time $t \in [0, T]$ which immediately imply the error estimate for the classical method.

In the sequel we use the notation $\xi(t) = U(t) - \Pi u(t)$, $\eta(t) = \Pi u(t) - u(t)$ and $e(t) = U(t) - u(t) = \xi(t) + \eta(t)$.

Lemma 3. *Let u satisfy (2). Then*

$$\|\eta(t)\| \leq Ch^{p+1}, \tag{19}$$

$$(u(t_{m-1} + s) - u^{m-1} - s\frac{\partial u}{\partial t}(t_{m-1} + s/2), w) \leq Cs^3\|w\| \quad \forall w \in S_h, \forall s \tag{20}$$

$$(\eta(t_{m-1} + s) - \eta^{m-1}, w) \leq Csh^{p+1}\|w\| \quad \forall w \in S_h, \forall s \tag{21}$$

Proof. The estimate (19) is standard estimate for L^2 projection approximation. The estimate (20) can be done similarly as in [4] and the last estimate (21) can be found in [1]. □

Lemma 4. *Let $p > d/2$. Let $s \in (0, \tau]$. If $\|e(t)\| \leq h^{1+d/2}$ for $t \leq t_{m-1} + s$, then*

$$\sup_{t \in [0, t_{m-1}+s]} \|e(t)\|^2 \leq C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^4), \tag{22}$$

where the constant C_T is independent of h, τ, ε.

Proof. Multiplying (5) for $t = t_{m-1} + s/2$ by s, subtracting from (13) and adding several terms we get

$$\begin{aligned}
&\Big(\xi(s) - \xi^{m-1}, w) + \frac{s\varepsilon}{2} A_h(\xi(s) + \xi^{m-1}), w\Big) \qquad (23)\\
&\le \left(s\frac{\partial u}{\partial t}(t_{m-1} + s/2) - u(s) + u^{m-1}, w\right)\\
&+s\left(b_h(u(t_{m-1}+s/2), w) - b_h\left(\frac{u(s)+u^{m-1}}{2}, w\right)\right) + (\eta(s) - \eta^{m-1}, w)\\
&+s\left(b_h\left(\frac{u(s)+u^{m-1}}{2}, w\right) - b_h\left(\frac{U(s)+U^{m-1}}{2}, w\right)\right) - \frac{s\varepsilon}{2} A_h(\eta(s)+\eta^{m-1}), w)\\
&+s\left(A_h(u(t_{m-1}+s/2), w) - A_h\left(\frac{u(s)+u^{m-1}}{2}, w\right)\right).
\end{aligned}$$

Setting $w = \xi(s) + \xi^{m-1}$ and using Lemmas 1–3 to estimate the right–hand side we get

$$\begin{aligned}
&\|\xi(s)\|^2 - \|\xi^{m-1}\|^2\\
&\qquad \le Cs\left(1 + \frac{\|e(s) + e^{m-1}\|_\infty^2}{h^2}\right)(\varepsilon h^{2p} + h^{2p+1} + \tau^2 + \|\xi(s)\|^2 + \|\xi^{m-1}\|^2).
\end{aligned}$$

Using the assumptions we can get rid of the unpleasant term $\|e(s)+e^{m-1}\|_\infty^2/h^2$ and by standard Gronwall lemma we can finish the proof. □

We are ready to present the main result.

Theorem 5. *Let $p > 1 + d/2$. Let $h_1, \tau_1 > 0$ are such that*

$$C_T^2(h_1^{2p+1} + \varepsilon h_1^{2p} + \tau_1^4) \le \frac{1}{2} h_1^{2+d}. \qquad (24)$$

Let τ_1 is sufficiently small to guarantee the existence and continuity of the continuated discrete solution. Then for all $h \in (0, h_1)$ and $\tau \in (0, \tau_1)$ we get

$$\sup_{t\in[0,T]} \|e(t)\|^2 \le C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^4), \qquad (25)$$

where the constant C_T is independent of h, τ, ε.

Proof. We will follow the idea of continuous mathematical induction from [6]. For time $t = 0$ it is easy to see that the error estimate holds true, because the error is in fact the error of L^2 projection in initial data, which is sufficiently small under the assumptions of the theorem. Let as assume that the error estimate holds true on the interval $[0, s]$. According to the assumption (24) we can see that the error can be estimated by $\|e(t)\| \le \frac{1}{2} h^{1+d/2}$, $t \in [0, s]$. Since the error $e(t)$ is continuous (even uniformly continuous) we know that there exists some $\delta > 0$ such that $\|e(t)\| \leq h^{1+d/2}$, $t \in [0, s+\delta]$ and we can see that it is possible to use Lemma 4 even on the interval $[0, s+\delta]$, which guarantees the error estimate on $[0, s+\delta]$. Since the error is uniformly continuous, we have fixed $\delta > 0$ during the process and using the argument repeatedly we obtain the result. □

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