Jan Česenek; Miloslav Feistauer
Interaction of compressible flow with an airfoil

In: Jan Chleboun and Petr Přikryl and Karel Segeth and Jakub Šístek (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 6-11, 2010. Institute of Mathematics AS CR, Prague, 2010. pp. 35--41.

Persistent URL: http://dml.cz/dmlcz/702738

## INTERACTION OF COMPRESSIBLE FLOW WITH AN AIRFOIL*

Jan Česenek, Miloslav Feistauer

**Abstract**

The paper is concerned with the numerical solution of interaction of compressible flow and a vibrating airfoil with two degrees of freedom, which can rotate around an elastic axis and oscillate in the vertical direction. Compressible flow is described by the Navier-Stokes equations written in the ALE form. This system is discretized by the semi-implicit discontinuous Galerkin finite element method (DGFEM) and coupled with the solution of ordinary differential equations describing the airfoil motion. Computational results showing the flow induced airfoil vibrations are presented.

### 1 Formulation of the continuous problem

We consider 2D compressible viscous flow in a bounded domain $\Omega(t) \subset R^2$ depending on time $t \in [0,T]$. We assume that the boundary $\partial\Omega(t)$ of $\Omega(t)$ consists of three disjoint parts: $\partial\Omega(t) = \Gamma_I \cup \Gamma_O \cup \Gamma_W(t)$, where $\Gamma_I$ is inlet, $\Gamma_O$ is outlet and $\Gamma_W(t)$ is impermeable wall, whose part may move.

The time dependence of the domain is taken into account with the aid of a regular one-to-one ALE mapping (cf. [4]) $\mathcal{A}_t : \Omega_0 \longrightarrow \Omega_t$, i.e. $\mathcal{A}_t : X \longmapsto x = x(X,t) = \mathcal{A}_t(X)$. We define the ALE velocity $\tilde{\boldsymbol{z}}(X,t) = \partial\mathcal{A}_t(X)/\partial t$, $\boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}^{-1}(x),t)$, $t \in [0,T]$, $X \in \Omega_0$, $x \in \Omega_t$, and the ALE derivative of a function $f = f(x,t)$ defined for $x \in \Omega_t$ and $t \in (0,T)$: $D^A f(x,t)/Dt = \partial\tilde{f}(X,t)/\partial t$, where $\tilde{f}(X,t) = f(\mathcal{A}_t(X),t)$, $X \in \Omega_0$.

The system describing compressible flow consisting of the continuity equation, the Navier-Stokes equations and the energy equation (see, e.g. [2]) can be written in the ALE form

$$\frac{D^A\boldsymbol{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w}\,\mathrm{div}\boldsymbol{z} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w},\nabla\boldsymbol{w})}{\partial x_s}, \tag{1}$$

where for $i,j = 1,2$ we have

$$\boldsymbol{w} = (w_1,\dots,w_4)^T = (\rho,\rho v_1,\rho v_2,E)^T \in I\!R^4, \quad \boldsymbol{g}_i(\boldsymbol{w}) = \boldsymbol{f}_i(\boldsymbol{w}) - z_i\boldsymbol{w}, \tag{2}$$
$$\boldsymbol{f}_i(\boldsymbol{w}) = (f_{i1},\cdots,f_{i4})^T = (\rho v_i, \rho v_1 v_i + \delta_{1i}\,p, \rho v_2 v_i + \delta_{2i}\,p, (E+p)v_i)^T,$$
$$\boldsymbol{R}_i(\boldsymbol{w},\nabla\boldsymbol{w}) = (R_{i1},\dots,R_{i4})^T = \left(0, \tau^V_{i1}, \tau^V_{i2}, \tau^V_{i1}\,v_1 + \tau^V_{i2}\,v_2 + k\partial\theta/\partial x_i\right)^T,$$
$$\tau^V_{ij} = (-2\,\mathrm{div}\boldsymbol{v}/3\,\delta_{ij} + 2\,d_{ij}(\boldsymbol{v}))/Re, \;\; d_{ij}(\boldsymbol{v}) = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2.$$

*The research of J. Česenek was supported by the Grant No. 12810 of the Grant Agency of the Charles University Prague. The research of M. Feistauer is a part of the research project MSM 0021620839 of the Ministry of Education of the Czech Republic. It was also partly supported by the grant No. 201/08/0012 of the Czech Science Foundation.

We use the following notation: $\rho$ - density, $p$ - pressure, $E$ - total energy, $\boldsymbol{v} = (v_1, v_2)$ - velocity, $\theta$ - absolute temperature, $\gamma > 1$ - Poisson adiabatic constant, $c_v > 0$ - specific heat at constant volume, $Re$ - the Reynolds number, $k$ - heat conduction. The vector-valued function $\boldsymbol{w}$ is called the state vector, the functions $\boldsymbol{f}_i$ are the so-called inviscid fluxes and $\boldsymbol{R}_i$ represent viscous terms. The above system is completed by the thermodynamical relations

$$p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2),\ \theta = \left(E/\rho - |\boldsymbol{v}|^2/2\right)/c_v$$

and equipped with the initial condition $\boldsymbol{w}(x, 0) = \boldsymbol{w}^0(x),\ x \in \Omega_0$, and the following boundary conditions:

$$\rho = \rho_D,\ \boldsymbol{v} = \boldsymbol{v}_D,\ \sum_{i,j=1}^{2} \tau_{ij}^V n_i v_j + k\frac{\partial \theta}{\partial n} = 0 \ \text{ on } \Gamma_I,$$

$$\boldsymbol{v}|_{\Gamma_{W_t}} = \boldsymbol{z}_D \ - \ \text{velocity of a moving wall},\ \partial\theta/\partial n = 0 \text{ on } \Gamma_{W_t},$$

$$\sum_{i=1}^{2} \tau_{ij}^V n_i = 0,\ j = 1,\, 2,\ \partial\theta/\partial n = 0 \text{ on} \Gamma_O,$$

with given data $\boldsymbol{w}^0$, $\rho_D$, $\boldsymbol{v}_D$, $\boldsymbol{z}_D$.

The terms $\boldsymbol{R}_s$ and $\boldsymbol{f}_s$ satisfy the relations

$$\boldsymbol{R}_s(\boldsymbol{w}, \nabla\boldsymbol{w}) = \sum_{k=1}^{2} \mathbf{K}_{s,k}(\boldsymbol{w})\frac{\partial \boldsymbol{w}}{\partial x_k}, \quad \boldsymbol{f}_s(\boldsymbol{w}) = \mathbf{A}_s(\boldsymbol{w})\boldsymbol{w}, \tag{3}$$

where $\mathbf{K}_{s,k}(\boldsymbol{w}) \in R^{4\times 4}$ and $\mathbf{A}_s$ is the Jacobian matrix of $\boldsymbol{f}_s$.

## 2 Discretization

### 2.1 Discontinuous Galerkin space discretization

By $\Omega_h(t)$ we denote polygonal approximation of the domain $\Omega(t)$. Let $\mathcal{T}_h(t) = \{K_i\}_{i\in I(t)}$ be a triangulation of the domain $\Omega_h(t)$ formed by a finite number of closed triangles $K_i$ with mutually disjoint interiors. We set $h_K = diam(K)$ as the diameter of $K$, $h(t) = \max_{K\in\mathcal{T}_h(t)} h_K$, $|\,K|$ is the Lebesgue measure of $K$. All elements of $\mathcal{T}_h(t) = \{K_i\}_{i\in I(t)}$ will be numbered so that $I(t) \subset Z^+ = \{0, 1, 2, 3, ...\}$ is a suitable index set. If two elements have a common face, than we call them neighbours and put $\Gamma_{ij} = \Gamma_{ji} = \partial K_i \cap \partial K_j$. For each $i \in I(t)$ we define the index set $s(i)(t) = \{j \in I(t); K_j \text{ is a neighbour of } K_i\}$. The boundary $\partial\Omega_h(t)$ is formed by a finite number of sides of elements $K_i$ adjacent to $\partial\Omega_h(t)$. We denote all these boundary sides by $S_j$, where $j \in I_b(t) \subset Z^- = \{-1, -2, -3, ...\}$ and set $\gamma(i)(t) = \{j \in I_b(t); S_j \text{ is a side of } K_i\}, \Gamma_{ij} = S_j$ for $K_i \in \mathcal{T}_h(t)$ such that $S_j \subset \partial K_i, j \in I_b(t)$. For an element $K_i$, not containing any boundary side $S_j$, we set $\gamma(i)(t) = \emptyset$. Obviously $s(i)(t) \cap \gamma(i)(t) = \emptyset$ for all $i \in I(t)$. Moreover we define $S(i)(t) = s(i)(t) \cup \gamma(i)(t)$.

We shall look for an approximate solution of the problem in the space $\mathbf{S}_h(t) = \{v; v\,|_K \in P^r(K), \forall K \in \mathcal{T}_h(t)\}^4$, where $r \geq 0$ is an integer and $P^r(K)$ is the space of polynomials of degree at most $r$ on $K$. If $v \in \mathbf{S}$, then we use the notation $v|_{\Gamma_{ij}}$ and $v|_{\Gamma_{ji}}$ for the traces of $v$ on $\Gamma_{ij}$ from the side of the adjacent elements $K_i$ and $K_j$, respectively, $\langle v\rangle_{\Gamma_{ij}}$ for the average of traces of $v$ on the face $\Gamma_{ij}$ from the side of the adjacent elements and $[v]_{\Gamma_{ij}}$ the jump of $v$ on $\Gamma_{ij}$. By $\boldsymbol{n}_{ij}$ we denote the unit outer normal to the boundary of $K_i$ on $\Gamma_{ij}$.

For arbitrary $t \in [0,T]$ we can multiply the system by a test function $\varphi \in \mathbf{S}_h(t)$ integrate and sum over all $K_i \in \mathcal{T}_h(t)$, apply Green's theorem and introduce a numerical flux $\mathbf{H}$. Then we introduce the following forms (cf. [1]):

$$
\begin{aligned}
\tilde{b}_h(\boldsymbol{w},\boldsymbol{\varphi}_h) =& -\sum_{i\in I(t)}\int_{K_i}\sum_{s=1}^{2}\boldsymbol{g}_s(\boldsymbol{w})\frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\mathrm{d}x + \sum_{i\in I(t)}\sum_{i\in S(i)(t)}\int_{\Gamma_{ij}}\mathbf{H}(\boldsymbol{w}|_{\Gamma_{ij}},\boldsymbol{w}|_{\Gamma_{ji}},\boldsymbol{n}_{ij})\mathrm{d}S\\
\tilde{a}_h(\boldsymbol{w},\boldsymbol{\varphi}_h) =& -\sum_{i\in I(t)}\int_{K_i}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}_{s,k}(\boldsymbol{w})\frac{\partial \boldsymbol{w}}{\partial x_k}\cdot\frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\mathrm{d}x\\
&+\sum_{i\in I(t)}\sum_{\substack{j\in s(i)(t)\\ j<i}}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\left\langle\sum_{k=1}^{2}\mathbf{K}_{s,k}(\boldsymbol{w})\frac{\partial \boldsymbol{w}}{\partial x_k}\right\rangle(n_{ij})_s\cdot[\boldsymbol{\varphi}_h]\mathrm{d}S\\
&+\sum_{i\in I(t)}\sum_{j\in\gamma_D(i)(t)}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}_{s,k}(\boldsymbol{w})\frac{\partial \boldsymbol{w}}{\partial x_k}(n_{ij})_s\cdot\boldsymbol{\varphi}_h\mathrm{d}S\\
&+\Theta\sum_{i\in I(t)}\sum_{\substack{j\in s(i)(t)\\ j<i}}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\left\langle\sum_{k=1}^{2}\mathbf{K}^T_{k,s}(\boldsymbol{w})\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}\right\rangle(n_{ij})_s\cdot[\boldsymbol{w}]\mathrm{d}S\\
&+\Theta\sum_{i\in I(t)}\sum_{j\in\gamma_D(i)(t)}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}^T_{k,s}(\boldsymbol{w})\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}(n_{ij})_s\cdot\boldsymbol{w}\mathrm{d}S\\
J_h^\sigma(\boldsymbol{w},\boldsymbol{\varphi}_h) =& \sum_{i\in I(t)}\sum_{\substack{j\in s(i)(t)\\ j<i}}\int_{\Gamma_{ij}}\sigma[\boldsymbol{w}]\cdot[\boldsymbol{\varphi}_h]\mathrm{d}S + \sum_{i\in I(t)}\sum_{j\in\gamma_D(i)(t)}\int_{\Gamma_{ij}}\sigma\boldsymbol{w}\cdot\boldsymbol{\varphi}_h\mathrm{d}S\\
\tilde{l}_h(\boldsymbol{w},\boldsymbol{\varphi}_h) =& \Theta\sum_{i\in I(t)}\sum_{j\in\gamma_D(i)(t)}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}^T_{k,s}(\boldsymbol{w})\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}(n_{ij})_s\cdot\boldsymbol{w}_B\mathrm{d}S\\
&+\sum_{i\in I(t)}\sum_{j\in\gamma_D(i)(t)}\int_{\Gamma_{ij}}\sigma\boldsymbol{w}_B\cdot\boldsymbol{\varphi}_h\mathrm{d}S,
\end{aligned}
$$

where $\sigma|_{\Gamma_{ij}} = \frac{C_W}{h(\Gamma_{ij})Re}$, $C_W > 0$ is a suitable sufficiently large constants and $\boldsymbol{w}_B$ is a boundary state defined by the Dirichlet boundary condition and extrapolation. By $(\cdot,\cdot)$ we denote the $L^2(\Omega(t_{k+1}))$-scalar product. We set $\Theta = -1$ or 0 or 1 and get the so-called nonsymmetric or incomplete or symmetric version of the viscous form. In practical computations we use $\Theta = 1$.

Now we can define the discrete problem: Find $\boldsymbol{w}_h(t) \in \mathbf{S}_h(t)$ such that

$$\begin{aligned} &\left(\frac{D^{\mathcal{A}}\boldsymbol{w}_h(t)}{Dt}, \boldsymbol{\varphi}_h\right) - (\mathrm{div}\boldsymbol{z}(t)\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) + \tilde{b}_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) + \tilde{a}_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) \\ + \;\; & J_h^{\sigma}(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) = \tilde{l}_h(\boldsymbol{\varphi}_h) \quad \forall \boldsymbol{\varphi}_h \in \mathbf{S}_h(t), \quad \forall t \in (0, T), \\ & \boldsymbol{w}_h(0) = \boldsymbol{w}_h^0. \end{aligned}$$

where $\boldsymbol{w}_h^0$ is the $\mathbf{S}_h(0)$-approximation of $\boldsymbol{w}^0$. It means that

$$\left(\boldsymbol{w}_h^0, \boldsymbol{\varphi}_h\right) = \left(\boldsymbol{w}^0, \boldsymbol{\varphi}_h\right) \quad \forall \boldsymbol{\varphi}_h \in \mathbf{S}_h(0).$$

**2.2 Time discretization**

Let us consider a partition $0 = t_0 < t_1 < ... < t_M$ of the interval $[0, T]$, $t_k = k\tau$, $\tau > 0$. We use the approximation $\boldsymbol{w}_h(t_l) \approx \boldsymbol{w}_h^l$, defined in $\Omega_h(t_l)$. Then we set $\hat{\boldsymbol{w}}_h^k(x) = \boldsymbol{w}_h^k(\mathcal{A}_{t_k}(\mathcal{A}_{t_{k+1}}^{-1}(x)))$, $\quad x \in \Omega_h(t_{k+1})$, and approximate the ALE-derivative using the first order backward difference:

$$\left(\frac{D^{\mathcal{A}}\boldsymbol{w}_h(t_{k+1})}{Dt}, \boldsymbol{\varphi}_h\right) \approx \left(\frac{\boldsymbol{w}_h^{k+1} - \hat{\boldsymbol{w}}_h^k}{\tau}, \boldsymbol{\varphi}_h\right).$$

Since the terms $\tilde{a}_h$ and $\tilde{b}_h$ are nonlinear, we shall linearized them. For $\tilde{b}_h$ we use the property (3) of $\boldsymbol{f}_s$ and the definition of $\boldsymbol{g}_s$. We get the approximation

$$\sum_{i \in I(t)} \int_{K_i} \sum_{s=1}^{2} \boldsymbol{g}_s(\boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} \mathrm{d}x \approx \sigma_1 = \sum_{i \in I(t_{k+1})} \int_{K_i} \sum_{s=1}^{2} \left(\mathbf{A}_s(\hat{\boldsymbol{w}}_h^k) - z_s \mathbf{I}\right) \boldsymbol{w}_h^{k+1} \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} \, \mathrm{d}x.$$

Now let us set $\mathbf{P}(\boldsymbol{w}, \boldsymbol{n}) := \sum_{s=1}^{2} \left(\mathbf{A}_s(\boldsymbol{w}) - z_s \mathbf{I}\right) n_s$, $(\boldsymbol{n} = (n_1, n_2), n_1^2 + n_2^2 = 1)$. We have $\sum_{s=1}^{2} \boldsymbol{g}_s(\boldsymbol{w}) n_s = \mathbf{P}(\boldsymbol{w}, \boldsymbol{n})\boldsymbol{w}$. It is possible to show that the matrix $\mathbf{P}$ is diagonalizable: $\mathbf{P} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$, where $\mathbf{T}$ is a nonsingular matrix, $\mathbf{D} = \mathrm{diag}(\lambda_1, ..., \lambda_4)$ is a diagonal matrix and $\lambda_i$ are the eigenvalues of $\mathbf{P}$. Then we can define the "positive" and "negative" parts of the matrix $\mathbf{P}$: $\mathbf{P}^{\pm} = \mathbf{T}\mathbf{D}^{\pm}\mathbf{T}^{-1}$, where $\mathbf{D}^{\pm} = \mathrm{diag}(\lambda_1^{\pm}, ..., \lambda_4^{\pm})$ and $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = \min(\lambda, 0)$. Using this concept, we introduce the so-called Vijayasundaram numerical flux

$$\mathbf{H}_V(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{n}) \;\; = \;\; \mathbf{P}^+\left(\frac{\boldsymbol{w}_1 + \boldsymbol{w}_2}{2}, \boldsymbol{n}\right)\boldsymbol{w}_1 + \mathbf{P}^-\left(\frac{\boldsymbol{w}_1 + \boldsymbol{w}_2}{2}, \boldsymbol{n}\right)\boldsymbol{w}_2.$$

Then we can approximate integrals over faces in the following way:

$$\begin{aligned} &\sum_{i \in I(t)} \sum_{j \in S(i)(t)} \int_{\Gamma_{ij}} \mathbf{H}(\boldsymbol{w}|_{\Gamma_{ij}}, \boldsymbol{w}|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}) \, \mathrm{d}S \;\approx\; \sigma_2 := \\ &\sum_{i \in I(t_{k+1})} \sum_{j \in S(i)(t_{k+1})} \int_{\Gamma_{ij}} \mathbf{P}^+\left(\frac{\hat{\boldsymbol{w}}_h^k|_{\Gamma_{ij}} + \hat{\boldsymbol{w}}_h^k|_{\Gamma_{ji}}}{2}, \boldsymbol{n}_{ij}\right) \boldsymbol{w}_h^{k+1}|_{\Gamma_{ij}} \cdot \boldsymbol{\varphi}_h \, \mathrm{d}S \\ + \;\; &\sum_{i \in I(t_{k+1})} \sum_{j \in S(i)(t_{k+1})} \int_{\Gamma_{ij}} \mathbf{P}^-\left(\frac{\hat{\boldsymbol{w}}_h^k|_{\Gamma_{ij}} + \hat{\boldsymbol{w}}_h^k|_{\Gamma_{ji}}}{2}, \boldsymbol{n}_{ij}\right) \boldsymbol{w}_h^{k+1}|_{\Gamma_{ji}} \cdot \boldsymbol{\varphi}_h \mathrm{d}S \end{aligned}$$

and define the form $b_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = -\sigma_1 + \sigma_2$.

Using (3), we linearize viscous terms:

$$a_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}\boldsymbol{\varphi}_h) = -\sum_{i\in I(t_{k+1})}\int_{K_i}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}_{s,k}(\hat{\boldsymbol{w}}_h^k)\frac{\partial \boldsymbol{w}_h^{k+1}}{\partial x_k}\cdot\frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\mathrm{d}x$$

$$+ \sum_{i\in I(t_{k+1})}\sum_{\substack{j\in s(i)(t_{k+1})\\ j<i}}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\left\langle\sum_{k=1}^{2}\mathbf{K}_{s,k}(\hat{\boldsymbol{w}}_h^k)\frac{\partial \boldsymbol{w}_h^{k+1}}{\partial x_k}\right\rangle(n_{ij})_s\cdot[\boldsymbol{\varphi}_h]\mathrm{d}S$$

$$+ \sum_{i\in I(t_{k+1})}\sum_{j\in\gamma_D(i)(t_{k+1})}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}_{s,k}(\hat{\boldsymbol{w}}_h^k)\frac{\partial \boldsymbol{w}_h^{k+1}}{\partial x_k}(n_{ij})_s\cdot\boldsymbol{\varphi}_h\mathrm{d}S$$

$$+ \Theta\sum_{i\in I(t_{k+1})}\sum_{\substack{j\in s(i)(t_{k+1})\\ j<i}}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\left\langle\sum_{k=1}^{2}\mathbf{K}_{k,s}^T(\hat{\boldsymbol{w}}_h^k)\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}\right\rangle(n_{ij})_s\cdot[\boldsymbol{w}_h^{k+1}]\mathrm{d}S$$

$$+ \Theta\sum_{i\in I(t_{k+1})}\sum_{j\in\gamma_D(i)(t_{k+1})}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}_{k,s}^T(\hat{\boldsymbol{w}}_h^k)\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}(n_{ij})_s\cdot\boldsymbol{w}_h^{k+1}\mathrm{d}S,$$

and the right-hand side form:

$$l_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{\varphi}_h) = \Theta\sum_{i\in I(t_{k+1})}\sum_{j\in\gamma_D(i)(t_{k+1})}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\sum_{k=1}^{2}\mathbf{K}_{k,s}^T(\hat{\boldsymbol{w}}_h^k)\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}(n_{ij})_s\cdot\boldsymbol{w}_B^{k+1}\mathrm{d}S$$

$$+\sum_{i\in I(t_{k+1})}\sum_{j\in\gamma_D(i)(t_{k+1})}\int_{\Gamma_{ij}}\frac{C_W}{h(\Gamma_{ij})Re}\boldsymbol{w}_B^{k+1}\cdot\boldsymbol{\varphi}_h\mathrm{d}S$$

All these considerations lead us to the following semi-implicit scheme: For $k = 0, 1, ...$ find $\boldsymbol{w}_h^{k+1} \in \mathbf{S}_h(t_{k+1})$ such that

$$\left(\frac{\boldsymbol{w}_h^{k+1}-\hat{\boldsymbol{w}}_h^k}{\tau}, \boldsymbol{\varphi}_h\right) - \left(\mathrm{div}\boldsymbol{z}(t_{k+1})\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h\right) + b_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) \tag{4}$$
$$+a_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + J_h^\sigma(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = l_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{\varphi}_h) \quad \forall \boldsymbol{\varphi}_h \in \mathbf{S}_h(t_{k+1}).$$

## 3 Fluid-structure interaction

We shall simulate motion of a profile with two degrees of freedom: $H$ - displacement of the profile in the vertical direction and $\alpha$ - the rotation of the profile around the so-called elastic axis. The motion of the profile is described by the system of ordinary differential equations

$$\begin{aligned} m\ddot{H} + k_{HH}H + S_\alpha\ddot{\alpha} &= -L(t), \\ S_\alpha\ddot{H} + I_\alpha H + k_{\alpha\alpha}\alpha &= M(t), \end{aligned} \tag{5}$$

where we use the following notation: $m$ - mass of the airfoil, $L(t)$ - aerodynamic lift force, $M(t)$ - aerodynamic torsional moment, $S_\alpha$ - static moment of the airfoil

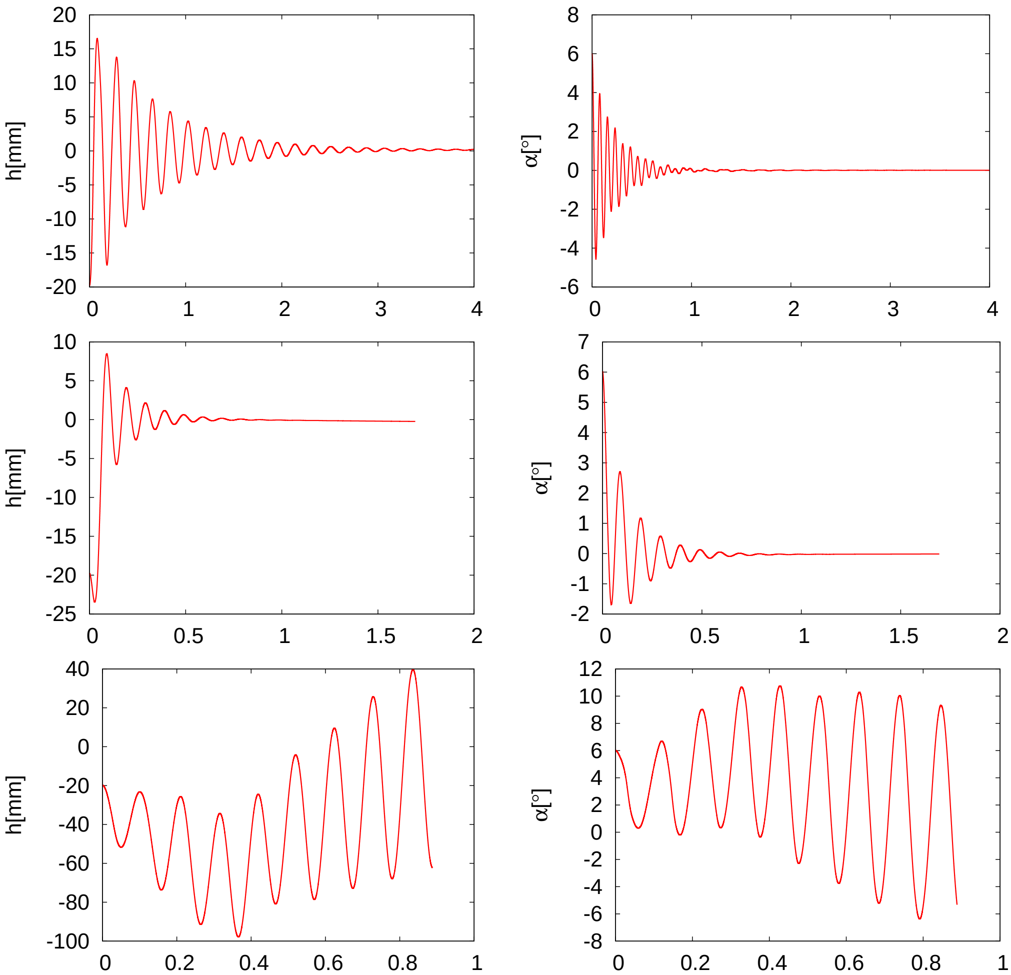


**Fig. 1:** *Displacement $H$ (left) and rotation angle $\alpha$ (right) of the airfoil in dependence on time for far-field velocity 10, 30 and 40 m/s.*

around the elastic axis, $I_\alpha$ - inertia moment of the airfoil around the elastic axis, $k_{HH}$ - bending stiffness, $k_{\alpha\alpha}$ - torsional stiffness. For the derivation of system (5) see, e.g. [5].

System (5) is transformed to a first-order system and solved by the fourth-order Runge-Kutta method together with the discrete flow problem (4). The ALE mapping is constructed on the new time level $t_{k+1}$ on the basis of the computed values $H(t_{k+1})$ and $\alpha(t_{k+1})$.

## 4 Numerical experiments

We perform numerical experiments with the following data and initial conditions: $m = 0.086622$ kg, $S_a = -0.000779673$ kg m, $I_a = 0.000487291$ kg m$^{-2}$, $k_{HH} = 105.109$ N/m, $k_{\alpha\alpha} = 3.696682$ Nm/rad, $l = 0.05$ m, $c = 0.3$ m, far-field density $\rho = 1.225$ kg m$^{-3}$, $H(0) = -20$mm, $\alpha(0) = 6°$, $\dot{H}(0) = \dot{\alpha}(0) = 0$.

Figure 1 shows the displacement $H$ and the rotation angle $\alpha$ in dependence on time for the far-field velocity 10, 30 and 40 m/s. We see that for the velocities 10 and 30 m/s the vibrations are damped, but for the velocity 40 m/s we get the flutter instability when the vibration amplitudes are increasing in time. The monotonous increase and decrease of the average values of $H$ and $\alpha$, respectively, shows that the flutter is combined with a divergence instability in the presented example.

These results are qualitatively comparable with vibrations of the airfoil NACA 0012 induced by viscous incompressible flow, contained in [3]. For low far-field velocity the differences of the presented results and results from [3] are small, because the compressibility of the fluid is not significant. For the far-field velocity 40 m/s the qualitative behaviour of the vibrations (flutter combined with divergence) is comparable with the results in [3] obtained by the finite element method. The quantitative difference is already larger probably due to compressibility taken into account in the present paper.

**References**

[1] Dolejší, V.: Semi-implicit interior penalty discontinuous Galerkin methods for viscous compressible flows. Commun. Comput. Phys. **4** (2008), 231–274.

[2] Feistauer, M., Felcman, J., and Straškraba, I.: *Mathematical and computational methods for compressible flow.* Clarendon Press, Oxford, 2003.

[3] Honzátko, R., Horáček, J., Kozel, K., and Sváček, P.: Simulation of free airfoil vibrations in incompressible viscous flow - comparison of FEM and FVM. Appl. Math. Comput. (submitted).

[4] Nomura, T. and Hughes, T.J.R.: An arbitrary Lagrangian-Eulerian finite element method for interaction of fluid and a rigid body. Comput. Methods Appl. Mech. Engrg. **95** (1992), 115–138.

[5] Sváček, P., Feistauer, M., and Horáček, J.: Numerical simulation of flow induced airfoil vibrations with large amplitudes. J. of Fluids and Structures **23** (2007), 391–411.