Petr Harasim

On the worst scenario method: Application to uncertain nonlinear differential equations with numerical examples

In: Jan Chleboun and Petr Přikryl and Karel Segeth and Jakub Šístek (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 6-11, 2010. Institute of Mathematics AS CR, Prague, 2010. pp. 84--88.

Persistent URL: `http://dml.cz/dmlcz/702743`

ON THE WORST SCENARIO METHOD: APPLICATION TO UNCERTAIN NONLINEAR DIFFERENTIAL EQUATIONS WITH NUMERICAL EXAMPLES*

Petr Harasim

1 Introduction: The worst scenario method

A great many problems in science can be described and solved by means of suitable mathematical models. Nevertheless, since the input data of mathematical models is encumbered with various sorts of uncertainty, the output values are also uncertain. It is our goal to evaluate the uncertainty of output data if the uncertainty of input data is somehow specified.

The mentioned models are characterized by a state problem $\mathcal{P}(a, u)$, where a represents input data and u denotes a solution of the state problem, so called state solution. The state problem $\mathcal{P}(a, u)$ can be represented by a boundary value problem, for instance. We consider a state problem whose input data is uncertain. Thus, let $\mathcal{U}_{\mathrm{ad}}$ be a given set of admissible input data. Since the state solution u depends on the input parameter $a \in \mathcal{U}_{\mathrm{ad}}$, we obtain a set of state solutions. As a rule, we are concerned with a real-valued quantity of interest related to the state solution and represented by a criterion functional $\Phi = \Phi(a, u(a))$, generally directly dependent on a. Due to the uncertainty of the state solution, we obtain a set of values of the criterion functional.

There exists a number of approaches to treatments of uncertainty in mathematical models. The choice of an acceptable approach depends largely on the amount of available information about the input data. If only the set of admissible input data is known, we wish to derive the corresponding set of outputs. In engineering applications, mainly large values of the quantity of interest (e.g. temperature at a selected point of a heated body, or local mechanical stress at a point of a loaded body) are important. Therefore, we search for an input parameter $a^0 \in \mathcal{U}_{\mathrm{ad}}$ such that the quantity of interest is maximal, i.e. we search for the worst scenario.

More precisely, let the state problem $\mathcal{P}(a, u)$ be given, $a \in \mathcal{U}_{\mathrm{ad}} \subset U$, $u \in V$, where U and V are suitable Banach spaces, and let Φ be the criterion functional mentioned above. The goal is to solve the following worst scenario problem: Find $a^0 \in \mathcal{U}_{\mathrm{ad}}$ such that

$$a^0 = \arg\max_{a \in \mathcal{U}_{\mathrm{ad}}} \Phi(a, u(a)). \tag{1}$$

*The work was supported by the Academy of Science of the Czech Republic, Institutional Research Plan No. AV0Z 30860518. The author would like to thank Dr. J. Chleboun and Dr. P. Byczanski for their help during the work.

The existence of the solution to problem (1) can be proved via the convergence of the solutions to approximate worst scenario problems, see [2]. The approximate worst scenario problem is defined as follows: Find $a_h^{M0} \in \mathcal{U}_{\mathrm{ad}}^M$ such that

$$a_h^{M0} = \arg \max_{a^M \in \mathcal{U}_{\mathrm{ad}}^M} \Phi(a^M, u_h(a^M)), \tag{2}$$

where $\mathcal{U}_{\mathrm{ad}}^M \subset \mathcal{U}_{\mathrm{ad}}$ is a M-dimensional approximation of the admissible set $\mathcal{U}_{\mathrm{ad}}$, $u_h(a^M) \in V_h \subset V$ is the solution of the state problem in a finite-dimensional subspace V_h of space V (usually, we use a finite element space). This approach also provides a way to calculate, at least approximately, the worst scenario a^0 and the corresponding value $\Phi(a^0, u(a^0))$.

For a more detailed mathematical treatment of the worst scenario method, see, e.g., [1, 2, 3, 4, 5].

2 Application to a one dimensional nonlinear boundary value problem

2.1 Definition of the problem

We consider the state problem examined in [2] and motivated by a boundary value problem with an ordinary differential equation: Find $u \in H_0^1(0,1)$ such that

$$\int_0^1 a(u'^2)u'v'\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x \qquad \forall v \in H_0^1(0,1), \tag{3}$$

where $H_0^1(0,1)$ is the usual Sobolev space, the function $a \in \mathcal{U}_{\mathrm{ad}}$ is an admissible coefficient, $f \in L^2(0,1)$. Let the admissible set $\mathcal{U}_{\mathrm{ad}}$ be a set of Lipschitz continuous functions a defined on $\mathbb{R}_0^+$ (nonnegative real numbers) and such that

$$\begin{aligned} 0 \le \frac{\mathrm{d}a}{\mathrm{d}x} \le & C_{\mathrm{L}} && \text{a.e. in } [0, x_C], \\ a(x) = & a(x_{\mathrm{C}}) && \text{for} \quad x \ge x_{\mathrm{C}}, \\ 0 < a_{\min} \le a(x) \le & a_{\max} && \forall x \in \mathbb{R}_0^+, \end{aligned}$$

where $C_{\mathrm{L}},\ x_{\mathrm{C}},\ a_{\min},\ a_{\max}$ are positive constants such that the admissible set is not empty.

Further, let T_j, $j \in \{1, \dots, M\}$, be equally spaced points in $[0, x_C]$, $T_1 = 0$ and $T_M = x_C$. We define the set $\mathcal{U}_{\mathrm{ad}}^M \subset \mathcal{U}_{\mathrm{ad}}$ of functions $a \in \mathcal{U}_{\mathrm{ad}}$ such that $a|_{[T_j, T_{j+1}]} \in P_1([T_j, T_{j+1}])$, $j \in \{1, \dots, M-1\}$, where $P_1([T_j, T_{j+1}])$ denotes the linear polynomials on the interval $[T_j, T_{j+1}]$. Moreover, we introduce equally spaced points $x_0 = 0 < x_1 < \dots < x_{N+1} = 1$ into interval $[0,1]$ and define $V_h \subset H_0^1(0,1)$, the space of functions continuous on $[0,1]$, linear on the interval $[x_i, x_{i+1}]$, $i = 0, \dots, N$, and with vanishing value at 0 and 1.

2.2 Algorithm and numerical results

In the following section, we show a procedure to find, at least approximately, a solution of problem (2), and present some numerical results. The computations were performed in MATLAB.

At first, we set $\Psi(a) = \Phi(a, u(a))$, so that we will examine a-dependent functional Ψ defined on $\mathcal{U}_{\mathrm{ad}}^M$. Furthermore, the finite-dimensional admissible set $\mathcal{U}_{\mathrm{ad}}^M$ can be identified with a compact subset $\widehat{\mathcal{U}}_{\mathrm{ad}}^M \subset \mathbb{R}^M$, if we define

$$\begin{aligned}\widehat{\mathcal{U}}_{\mathrm{ad}}^M = \{\alpha \in \mathbb{R}^M : \exists a \in \mathcal{U}_{\mathrm{ad}}^M \quad \alpha &= (\alpha_1, \ldots, \alpha_M) \\ &= (a(x_1), \ldots, a(x_M))\},\end{aligned}$$

see also [1]. In this sense, the functional Ψ is, as a matter of fact, a real function $\widehat{\Psi} = \widehat{\Psi}(\alpha)$, where $\alpha = (\alpha_1, \ldots, \alpha_M) \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M$. To obtain the value of function $\widehat{\Psi}$ at any point $\alpha \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M$, it is necessary to solve the following nonlinear problem (a finite-dimensional analogy to (3)): Find $u_h \in V_h$ such that

$$\int_0^1 a(u_h'^2) u_h' v' \mathrm{d}x = \int_0^1 f v \, \mathrm{d}x \qquad \forall v \in V_h, \tag{4}$$

where $a \in \mathcal{U}_{\mathrm{ad}}^M$, $a(x_i) = \alpha_i$, $i = 1, \ldots, M$. An approximation of the solution to problem (4) is obtained by using the Kachanov method, that is, by means of a (finite) sequence of the solutions to linearized problems, more detailed treatment can be found, e.g., in [3]. Subsequently, the criterion functional Φ is evaluated. The ultimate goal is to solve the following global optimization problem arising from (2): Find $\alpha^0 \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M$ such that

$$\alpha^0 = \arg \max_{\alpha \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M} \widehat{\Psi}(\alpha).$$

To find the element α^0 at least approximately, we use the Nelder-Mead simplex method. This method is implemented by the standard MATLAB function *fminsearch* (this algorithm requires to enter an initial point). However, to be able to solve our global optimization problem by the unconstrained optimization routine *fminsearch*, we establish a transformation $T : \mathbb{R}^M \to \widehat{\mathcal{U}}_{\mathrm{ad}}^M$ and search for the maximum of the composite function $\widehat{\Psi} \circ T : \mathbb{R}^M \to \mathbb{R}$. In the concrete, for $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$ we obtain the corresponding value $T(x) = \alpha = (\alpha_1, \ldots, \alpha_M) \in \widehat{\mathcal{U}}_{\mathrm{ad}}^M$ as follows: For the first component of α we define

$$\alpha_1 = a_{\min} + \frac{(a_{\max} - a_{\min})(\frac{\pi}{2} + \arctan x_1)}{\pi},$$

for α_i, $i = 2, \ldots, M$, we define

$$\alpha_i = \alpha_{i-1} + \frac{K(\frac{\pi}{2} + \arctan x_i)}{\pi},$$

where $K = \min\{\frac{C_L x_C}{M-1}, a_{\max} - \alpha_{i-1}\}$.

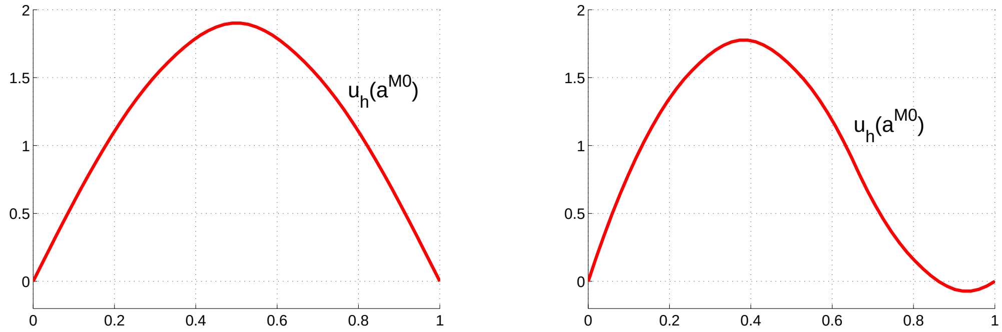

Fig. 1: *The state solution of the problem (4) with the parameter* a^{M0} *and the right-hand side* f_1 *(on the left) and* f_2 *(on the right).*

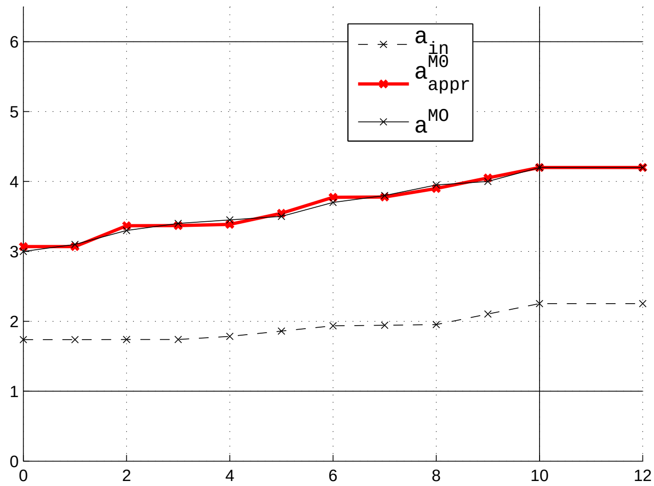

Fig. 2: *The approximation* a^{M0}_{appr} *of the searched parameter* a^{M0} *for the right-hand side* f_1 *and a given initial point* $\alpha_{\text{in}} \in \widehat{\mathcal{U}}^M_{\text{ad}}$ *corresponding to a parameter* $a_{\text{in}} \in \mathcal{U}^M_{\text{ad}}$ $(\widehat{\Psi}(\alpha_{\text{in}}) = -1.2828 \times 10^6, \widehat{\Psi}(\alpha^{M0}_{\text{appr}}) = -0.86 \times 10^{-2})$.

Now, we present concrete numerical examples. Let the parameters of admissible set $\mathcal{U}_{\text{ad}}$ be: $a_{\min} = 1$, $a_{\max} = 6$, $C_L = 0.3$, and $x_C = 10$. Let the dimension of $\mathcal{U}^M_{\text{ad}}$ be $M = 11$ and the dimension of the finite element space V_h be $N = 50$. We solve the state problem (4) with two different right-hand sides f_1 and f_2. Concretely, $f_1(x) = 300x(1-x)$, and

$$f_2(x) = \begin{cases} 100 & \text{for} \quad 0 \le x \le \frac{2}{3} \\ -100 & \text{for} \quad \frac{2}{3} < x \le 1. \end{cases}$$

The worst scenario problem (2) is solved with the following criterion functional:

$$\Phi(a, u(a)) = -10^6 \int_0^1 [u(a) - u_h(a^{M0})]^2 \, \mathrm{d}x,$$

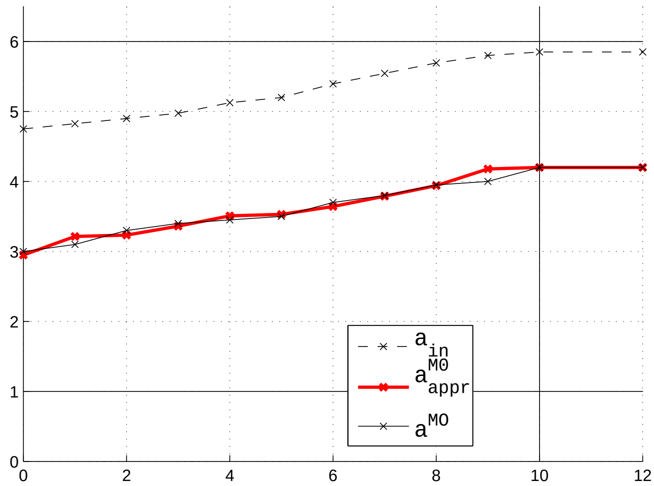

Fig. 3: *The approximation* a^{M0}_{appr} *of the searched parameter* a^{M0} *for the right-hand side* f_2 *and a given initial point* $\alpha_{\text{in}} \in \widehat{\mathcal{U}}^M_{\text{ad}}$ *corresponding to a parameter* $a_{\text{in}} \in \mathcal{U}^M_{\text{ad}}$ $(\widehat{\Psi}(\alpha_{\text{in}}) = -9.7035 \times 10^4, \widehat{\Psi}(\alpha^{M0}_{\text{appr}}) = -0.126 \times 10^{-1})$.

where $u_h(a^{M0}) \in V_h$ is the solution of problem (4) computed for a chosen (and afterwards searched) parameter a^{M0}, determined by the vector of nodal values $\alpha^0 = (3.00, 3.10, 3.30, 3.40, 3.45, 3.50, 3.70, 3.80, 3.95, 4.00, 4.20) \in \widehat{\mathcal{U}}^M_{\text{ad}}$. In this setting, the worst scenario problem turns into a parameter identification problem and, naturally, it holds $\widehat{\Psi}(\alpha^0) = 0$. The following figures present some numerical results.

References

[1] Chleboun, J.: Reliable solution for a 1D quasilinear elliptic equation with uncertain coefficients. J. Math. Anal. Appl. **234** (1999), 514–528.

[2] Harasim, P.: On the worst scenario method: A modified convergence theorem and its application to an uncertain differential equation. Appl. Math. **53** (2008), 583–598.

[3] Harasim, P.: On the worst scenario method: Application to a quasilinear elliptic 2D-problem with uncertain coefficiens. Appl. Math. Accepted

[4] Hlaváček, I.: Reliable solution of elliptic boundary value problems with respect to uncertain data. Nonlinear Anal. **30** (1997), 3879–3890.

[5] Hlaváček, I., Chleboun, J. and Babuška I.: *Uncertain input data problems and the worst scenario method*. Elsevier, Amsterdam, 2004.