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In: Jan Chleboun and Petr Přikryl and Karel Segeth and Jakub Šístek (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 6-11, 2010. Institute of Mathematics AS CR, Prague, 2010. pp. 101--106.

Persistent URL: `http://dml.cz/dmlcz/702746`

NUMERICAL MODELLING OF NEWTONIAN AND NON-NEWTONIAN FLUIDS FLOW IN THE BRANCHING CHANNEL BY FINITE VOLUME METHOD*

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1 Mathematical model

The governing system of the equations is the system of Navier-Stokes equations for incompressible fluids. This system for generalized Newtonian fluids can be written in the conservative form [1]:

$$\tilde{R}W_t + F_x^c + G_y^c = F_x^v + G_y^v, \quad \tilde{R} = \mathrm{diag}(0,1,1), \tag{1}$$

$$W = \begin{pmatrix} p \\ u \\ v \end{pmatrix}, \quad F^c = \begin{pmatrix} u \\ u^2+p \\ uv \end{pmatrix}, \quad G^c = \begin{pmatrix} v \\ uv \\ v^2+p \end{pmatrix}, \tag{2}$$

$$F^v = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \end{pmatrix}, \quad G^v = \begin{pmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \end{pmatrix}. \tag{3}$$

where $p = \frac{P}{\rho}$, P is the pressure, u, v are the components of the velocity vector, ρ is the constant density. The vector W is the vector of unknowns. The vectors F^c, G^c are inviscid physical fluxes and F^v, G^v are viscous physical fluxes. The viscous stress τ is defined as follows

$$\tau = 2\eta(\dot{\gamma})D, \quad \dot{\gamma} = \sqrt{\mathrm{tr}D^2}, \quad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{4}$$

where tensor D is the symmetric part of the velocity gradient and where i and j can take on the values x, y or 1, 2. The quantities x_1 and x_2 in the derivatives denote Cartesian coordinates x, y. Similarly v_1 and v_2 denote the velocity vector components u, v.

Newtonian and non-Newtonian fluids differ through the choice of the viscosity function. One of the simplest viscosity function is the power-law model [2]

$$\eta(\dot{\gamma}) = \nu\left(\sqrt{\mathrm{tr}D^2}\right)^r, \tag{5}$$

*This work was partly supported by grant GACR No. 101/09/1539, Research Plan MSM 684 077 0003 and Research Plan MSM 684 077 0010.

where ν is a constant, e.g. the kinematic viscosity for Newtonian fluids. The power r is the power-law index. The power-law model includes Newtonian fluids as a special case ($r = 0$). For $r > 0$ the power-law fluid is shear thickening (increasing viscosity with shear rate), while for $r < 0$ it is shear thinning (decreasing viscosity with shear rate).

2 Numerical solution

2.1 Steady computation

In this first part the steady state solution is considered. In such a case an artificial compressibility method can be applied, i.e. the continuity equation is completed by a term $\frac{1}{\beta^2}p_t$. In the non-dimensional form this yields

$$\tilde{R}_\beta W_t + F^c_x + G^c_y = \frac{1}{\mathrm{Re}}\,\epsilon\,(F^v_x + G^v_y), \quad \tilde{R}_\beta = \mathrm{diag}\left(\frac{1}{\beta^2}, 1, 1\right), \quad \beta \in \mathcal{R}^+ \qquad (6)$$

where in non-dimensional form F^c, G^c are inviscid physical fluxes and F^v, G^v are viscous physical fluxes. The symbol ϵ represents $\left(\sqrt{\mathrm{tr}D^2}\right)^r$. The symbol Re denotes Reynolds number and it's defined by the expression

$$\mathrm{Re} = \frac{U^* L^*}{\nu}, \qquad (7)$$

where U^*, L^* are the reference velocity and length, ν is the kinematic Newtonian viscosity. The parameter β has dimension of a speed. In the case of non-dimensional equations, β is then divided by a reference velocity U^*. This is usually an upstream velocity, which does not significantly differ from the maximum velocity in the flow field. Hence, in the case of non-dimensional equations, $\beta = 1$ is used in presented steady numerical simulations.

Eq. (6) is space discretized by the finite volume method[3], [5] and the arising system of ODEs is time discretized by the explicit multistage Runge-Kutta scheme of the second order of accuracy in the time

$$\begin{aligned} W_i^n &= W_i^{(0)} \\ W_i^{(r)} &= W_i^{(0)} - \alpha_{r-1}\Delta t \mathrm{Res}(W)_i^{(r-1)} \\ W_i^{n+1} &= W_i^{(m)} \qquad\qquad r = 1, \dots, m, \end{aligned} \qquad (8)$$

where $m = 3$, $\alpha_0 = \alpha_1 = 0.5, \alpha_2 = 1.0$, the steady residual $\mathrm{Res}(W)_i$ is defined by finite volume method as

$$\mathrm{Res}(W)_i = \frac{1}{\mu_i}\sum_{k=1}^{4}\left[\left(\overline{F}^c_k - \frac{1}{\mathrm{Re}}\epsilon\,\overline{F}^v_k\right)\Delta y_k - \left(\overline{G}^c_k - \frac{1}{\mathrm{Re}}\epsilon\,\overline{G}^v_k\right)\Delta x_k\right], \qquad (9)$$

where μ_i is the volume of the finite volume cell, $\mu_i = \int\int_{C_i} dx\, dy$. The symbols $\overline{F}^c_k, \overline{G}^c_k$ and $\overline{F}^v_k, \overline{G}^v_k$ denote the numerical approximation of the inviscid and viscous physical fluxes. The symbol Re is Reynolds number defined by (7). The symbol ϵ represents $\left(\sqrt{\mathrm{tr}D^2}\right)^r$, where for power r three values are choosed: $r = 0$ for Newtonian fluids, $r = 0.5$ for shear thickening fluids and $r = -0.5$ for shear thinning fluids.

2.2 Unsteady computation

The dual-time stepping method is used for the unsteady flows for Newtonian fluids. The principle of dual-time stepping method is following. The artificial time τ is introduced and the artificial compressibility method in the artificial time is applied. The system of Navier-Stokes equations is extended to unsteady flows by adding artificial time derivatives $\partial W/\partial\tau$ to all equations [4]

$$\tilde{R}_\beta W_\tau + \tilde{R} W_t + F_x^c + G_y^c = F_x^v + G_y^v \tag{10}$$

with matrices $\tilde{R}$, $\tilde{R}_\beta$ given by Eq. (1), (6). The vector of the variables W, the inviscid fluxes F^c, G^c and the viscous fluxes F^v, G^v are given by Eq. (2).

The derivatives with respect to the real time t are discretized using a three-point backward formula, it defines the form of unsteady residual

$$\tilde{R}_\beta \frac{W^{l+1} - W^l}{\Delta\tau} = -\tilde{R}\frac{3W^{l+1} - 4W^n + W^{n-1}}{2\Delta t} - \mathrm{Res}(W)^l = -\overline{\mathrm{Res}}(W)^{l+1}, \tag{11}$$

where $\Delta t = t^{n+1} - t^n$ and $\mathrm{Res}(W)$ is the steady residual defined as for steady computation, see Eq. (9). The symbol $\overline{\mathrm{Res}}(W)$ denotes unsteady residual. The superscript n denotes the real time index and the index l is associated with the pseudo-time. The integration in pseudo-time can be carried out by explicit multistage Runge-Kutta scheme.

The solution procedure is based on the assumption that the numerical solution at real time t^n is known. Setting $W_i^l = W_i^n, \forall i$, the iteration in l using explicit Runge-Kutta method are performed until the condition

$$\|\overline{\mathrm{Res}}(W)^l\|_{\mathrm{L}^2} = \sqrt{\sum_i \left(\frac{W_i^{l+1} - W_i^l}{\Delta\tau}\right)^2} \leq \epsilon \tag{12}$$

is satisfied for a chosen small positive number ϵ. The symbol $\overline{\mathrm{Res}}(W)^l$ stands for the vector formed by the collection of $\overline{\mathrm{Res}}(W)_i^l, \forall i$. Once the condition (12) is satisfied for a particular l, one sets $W_i^{n+1} = W_i^{l+1}, \forall i$. Then the index representing real-time level can be shifted one up. History of the convergence of unsteady residual in dual time from t^n to t^{n+1} is plotted in decadic logarithm.

The unsteady boundary conditions are defined as follows. In the inlet, in the solid wall and in one of the outlet part the steady boundary conditions are prescribed. In the second outlet part the unsteady boundary conditions are defined. The velocity is computed by the extrapolation from the domain. The pressure value is prescribed by the function

$$p_{21} = \frac{1}{4}\left(1 + \frac{1}{2}\sin(\omega t)\right), \tag{13}$$

where ω is the angular velocity defined as $\omega = 2\pi f$, where f is the frequency.

3 Numerical results

3.1 Two dimensional steady solution

In this section the steady numerical results of two dimensional incompressible laminar viscous flows for generalized Newtonian fluids are presented.

The following choices of the power-law index were used. For Newtonian fluid $r = 0$ is used. For shear thickening and shear thinning non-Newtonian fluid values $r = 0.5$ (shear thickening) and $r = -0.5$ (shear thinning) are used. The flow is computed through the branching channel. In the inlet the velocity is prescribed by the parabolic function. Reynolds number is equal to 400 for tested cases of the fluids.

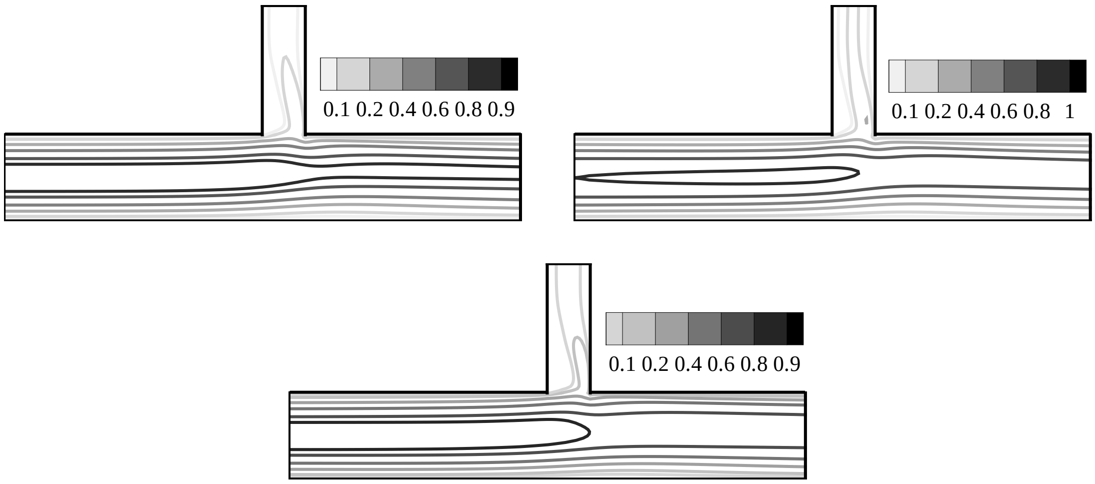

Fig. 1: *Velocity isolines of steady flows for generalized Newtonian fluids - a) Newtonian - b) shear thickening non-Newtonian - c) shear thinning non-Newtonian.*

In the Figures 1 the velocity isolines for 2D tested fluids are presented. One of the main differencies between Newtonian and non-Newtonian fluids is given by the size of the separation region.

In the Figure 2 the nondimensional axial velocity profile for steady fully developed flow of Newtonian, shear thickening and shear thinning fluids in 2D branching channel is shown.

3.2 Two dimensional unsteady numerical solution

In this section two dimensional unsteady numerical results for Newtonian fluid through the branching channel are presented. The dual-time stepping method are used. The unsteady boundary conditions were considered. As initial data the numerical solution of steady fully developed flow of Newtonian fluids in the branching channel were used. Reynolds number is 400. The frequency in the pressure function (13) is 2.

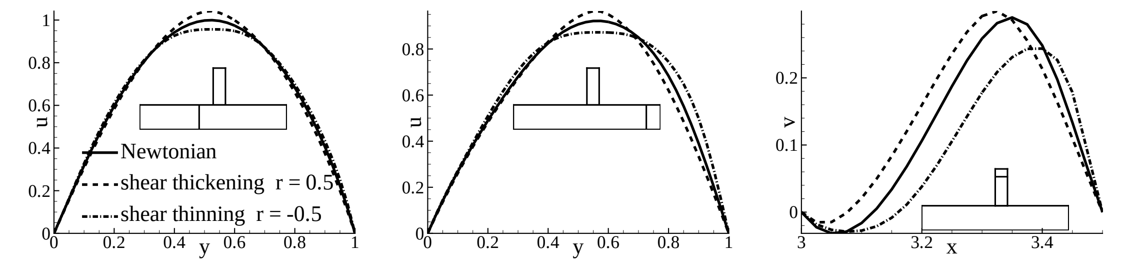

Fig. 2: *Nondimensional velocity profile for steady fully developed flow of generalized Newtonian fluids in the branching channel (the line legend in all three panels is the same).*

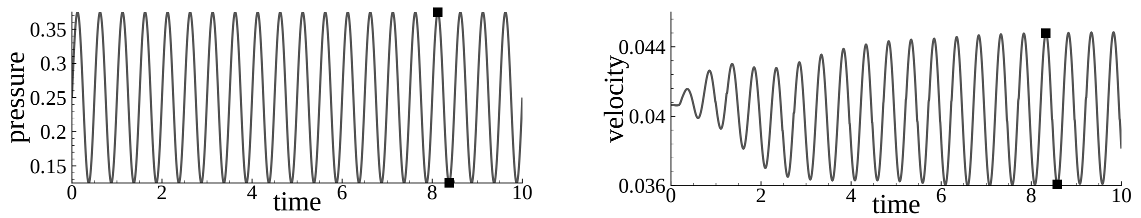

Fig. 3: *The graphs of the pressure and the velocity computed by the dual-time stepping method.*

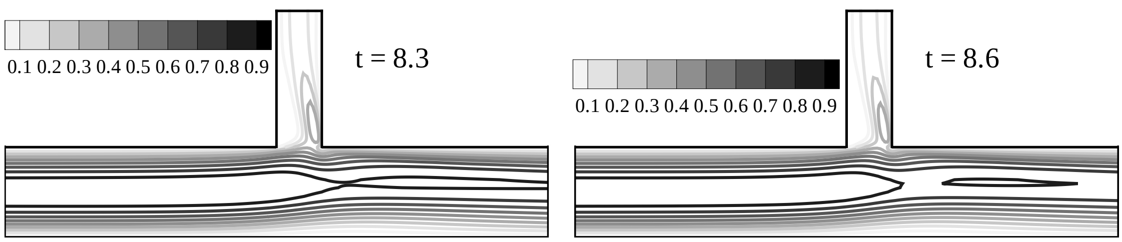

Fig. 4: *Velocity isolines of unsteady flows of Newtonian fluids - dual-time stepping method.*

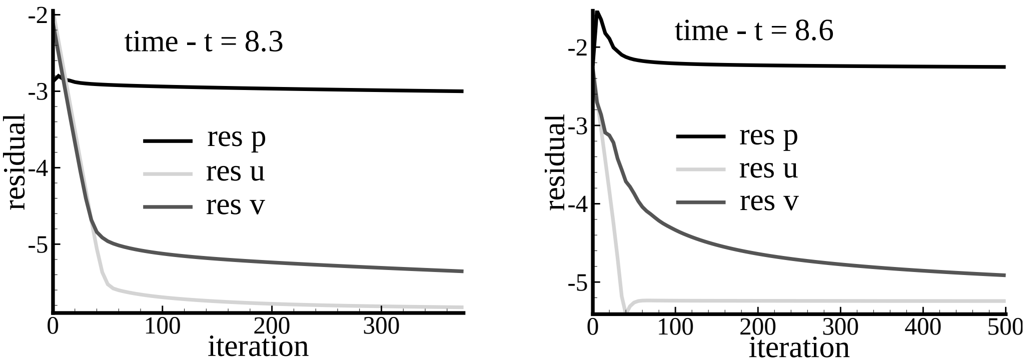

Fig. 5: *Decadic logarithm of the L^2 norm of the unsteady residual - dual-time stepping method.*

In Figure 3 the graphs of the pressure $p_{21}(t)$ and the velocity as the function of the time for Newtonian fluid are shown. By the square symbols the positions of the unsteady numerical results for dual-time stepping method shown in the Figure 4 are sketched. In the Figure 5 the decadic logarithm of the L^2 norm of unsteady residual by the dual-time stepping method is shown.

4 Conclusions

In this paper a finite volume solver for two and three dimensional incompressible laminar viscous flows in the branching channel was described. The numerical results obtained by this method for Newtonian and non-Newtonian (shear thickening and shear thinning) fluid flows were presented. For the generalized Newtonian fluids the power-law model was used. The explicit Runge-Kutta method was considered for numerical modelling. The convergence history confirms the robustness of the applied method.

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