Martin Kocurek
Insensitivity analysis of Markov chains


Persistent URL: http://dml.cz/dmlcz/702747

Terms of use:
© Institute of Mathematics AS CR, 2010

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library
http://project.dml.cz
INSENSITIVITY ANALYSIS OF MARKOV CHAINS

Martin Kocurek

Abstract

Sensitivity analysis of irreducible Markov chains considers an original Markov chain with transition probability matrix \( P \) and modified Markov chain with transition probability matrix \( \tilde{P} \). For their respective stationary probability vectors \( \pi, \tilde{\pi} \), some of the following characteristics are usually studied: \( \| \pi - \tilde{\pi} \|_p \) for asymptotical stability [3], \(| \pi_i - \tilde{\pi}_i |, \frac{| \pi_i - \tilde{\pi}_i |}{\pi_i} \) for componentwise stability or sensitivity[1]. For functional transition probabilities, \( P = P(t) \) and stationary probability vector \( \pi(t) \), derivatives are also used for studying sensitivity of some components of stationary distribution with respect to modifications of \( P \) [2].

In special cases, modifications of matrix \( P \) leave certain stationary probabilities unchanged. This paper studies some special cases which lead to this behavior of stationary probabilities.

1 Introduction

A Markov chain is a sequence of random variables \( X_1, X_2, X_3, \ldots \), with the Markov property, namely that, given the present state, the future and past states are independent. Formally,

\[
P(X_{n+1} = x | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_{n+1} = x | X_n = x_n),
\]

where the possible values of \( X_i \) form a countable state space \( S \) of the chain. Markov chains are often described by a directed graph, where the edges are labeled by the probabilities \( p_{ij} \) of moving from state \( i \) to the other state \( j \). These probabilities are called transition probabilities and together they form a transition probability matrix denoted by \( P \), with row sums equal to 1. We will study finite Markov chains (a finite chain has a finite state space \( S = \{x_1, \ldots, x_n\} \)). A state \( i \) has period \( p \) if any return to state \( i \) must occur in multiples of \( p \) time steps. Formally, the period of a state \( i \) is defined as \( p = \gcd\{k : P(X_k = i|X_0 = i) > 0\} \). If \( p = 1 \), then the state is said to be aperiodic i.e. returns to state \( i \) can occur at irregular times. Otherwise \( (p > 1) \), the state is said to be periodic with period \( p \). If all states are periodic with period \( p \), the chain is called \( p \)-cyclic.

Let us denote

\[
e = (1, \ldots, 1)^T, \quad e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T = (\delta_{i,j})_{j=1}^n, \quad i = 1, \ldots, n, \quad P = (P_{ij})_{i,j=1}^n.
\]

A Markov chain is called irreducible, if there exists a connection between every two states. That means, matrix \( P \) is irreducible. In this case, matrix \( P \) has a unique
eigenvalue 1 (which equals to spectral radius $\rho(P)$ of $P$) and unique left and right eigenvectors associated with this eigenvalue, $\pi = (\pi_1, \ldots, \pi_n)$ and $e$, so that

$$\pi P = \pi, \ Pe = e.$$ 

Vector $\pi$ is called *stationary probability vector*, we usually normalise this vector to $\pi e = \|\pi\|_1 = 1$; $i$-th component $\pi_i$ of $\pi$ shows, how often the chain “visits state $i$”,

$$\pi_i = \lim_{m \to \infty} \frac{|\{j; X_j = x_i, j = 1, \ldots, m\}|}{m}.$$ 

We will also use a different normalisation, $\pi_k = 1$ and in this case, the eigenvector will be denoted by $\pi_{(k)}$, so that $\pi_{(k)k} = 1$.

In the following, we will partition matrix $P$ and vector $\pi$ into subblocks,

$$\pi = (\pi^{(1)}, \ldots, \pi^{(N)}), \quad P = \begin{pmatrix} P_{11} & \cdots & P_{1N} \\ \vdots & \ddots & \vdots \\ P_{N1} & \cdots & P_{NN} \end{pmatrix}, \quad (1)$$

where $N$ is the number of subblocks in matrix $P$, $n_1, \ldots, n_N$ will be respective dimensions of subblocks. Conformally with partitioning of $P$ we shall partition vector $e = (e^{(1)}^T, \ldots, e^{(N)}^T)^T$, where $e^{(i)}$ is a vector $(1, \ldots, 1)^T$ with $n_i$ components.

As an example we will use a Markov chain with the following matrix:

$$P_c = \frac{1}{64} \begin{pmatrix} 62 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 62 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 60 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 62 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 63 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 62 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 60 & 2 & 0 & 0 & 0 & 0 \\ 62 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 62 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 & 62 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 60 \end{pmatrix}. \quad (2)$$

2 Normalisation $\pi_k = 1$

Normalisation $\pi_k = 1$ is useful for computing the eigenvector as a solution of a system of equations $\pi_{(k)} P = \pi_{(k)}$, or $P^T \pi_{(k)}^T = \pi_{(k)}^T$, $(I - P^T)\pi_{(k)}^T = 0$. By replacing an arbitrary equation with equation $\pi_{(k)} e_k = 1$, or equivalently, $e_k^T \pi_{(k)}^T = 1$, we obtain a system with better spectral properties, than when using condition $e^T \pi^T = 1$ [4].

When we use this normalization, we can state the following simple theorem.
**Theorem 1.** Let the state space of a Markov chain can be decomposed into three groups $S_1, \{x_k\} = S_2, S_3$, so that in oriented graph of the Markov chain each path from $S_1$ to $S_3$ contains a vertex $x_k$. Then no modifications of transition probabilities between states of $S_1$ affect components in $\pi(k)$ associated with states from $S_3$.

**Proof:** With given restrictions, the graph of the chain can be simplified into

\[
P = \begin{pmatrix}
    X & \ldots & X & X & 0 & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    X & \ldots & X & X & 0 & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    X & \ldots & X & X & X & \ldots & X \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    X & \ldots & X & X & X & \ldots & X
  \end{pmatrix}.
\]

At this picture, $S_1$ is denoted by 1, $x_k$ by k, $S_3$ by 3. It then follows that nonzero structure of $P$ is

\[
A^{(k)} = \begin{pmatrix}
    X & \ldots & X & X & X & \ldots & X \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    X & \ldots & X & X & X & \ldots & X \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & X & X & \ldots & X \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & X & X & \ldots & X
  \end{pmatrix},
\]

it is clearly reducible. Thus, no modifications of transition probabilities between states in $S_1$ (block 1,1 in $A^{(k)}$) will affect $k$-th, \ldots, $n$-th components of $\pi(k)$. 

109
Example: In example with $P_e$, we may draw an oriented graph:

![Graph](image)

We see that vertices 2, 3, 6, 7 are accessible only through vertex 1. Thus if we fix the first element of $\pi$, no modifications of transition probabilities between vertices 4, 5, 8, 9, 10, 11, 12, 13 will affect components no. 2, 3, 6, 7 in $\pi_{(1)}$.

3 Normalisation $\pi e = 1$

For a more usual normalisation $\pi e = 1$, let us first introduce a concept of lumpability.

**Definition:** Let us partition a transition probability matrix $P$ into blocks $(P_{ij})_{i,j=1}^N$ so that for every block $P_{ij}$ and vector $e^{(j)}$ of appropriate dimensions

$$P_{ij} e^{(j)} = \alpha_{ij} e^{(j)}$$

for some $\alpha_{ij} \in \mathbb{R}$. Then matrix $P$ is said to be lumpable.

**Theorem 2.** Let $P(t)$ be a perturbed transition probability matrix of an irreducible finite aperiodic Markov chain, whose state space divided into subsets $S_1, \ldots, S_{N+1}$, where states of $S_{N+1}$ are accessible only through $S_N$. Let perturbations depend on a variable $t$ and be restricted to lumpable submatrix of blocks $(P_{ij}(t))_{i,j=1}^{N-1}$. If for every $i = 1, \ldots, N - 1$ exists a column vector $x^{(i)}$ such that

$$P_{iN} = e^{(i)} : x^{(i)T},$$

then subblocks $\pi^{(N)}, \pi^{(N+1)}$ are independent of $t$.
Proof: From the assumption it follows that

\[ P_{ij} e^{(j)} = \alpha_{ij} e^{(j)}, \quad P_{i,N+1} = 0, \quad i, j = 1, \ldots, N-1. \quad (4) \]

We will prove the theorem by using a power method for computing \( \pi \). Assumptions guarantee the existence of a unique steady point – eigenvector \( \pi \) [4].

Let us choose a \( \pi^{(0)} = (\pi^{(0)}_1, \ldots, \pi^{(0)}_{N+1}) \), for \( l = 1, 2, \ldots \)

\[ \pi^{(l+1)} = \pi^{(l)} P. \]

a) At first we will show by induction, that for every \( l \) the \( \| \cdot \|_1 \)-norms of subvectors \( \pi^{(l)}_1, \ldots, \pi^{(l)}_{N+1} \) of \( \pi^{(l)} \) do not depend on \( t \). \( \pi^{(0)} \) does not depend on \( t \). The \( l_1 \)-norm of the \( j \)-th subvector, \( j = 1, \ldots, N-1 \), in the \( (l+1) \)-th iteration is

\[ \| \pi^{(l+1)}_j \|_1 = \pi^{(l)}_j P_{*,j} e^{(j)} = \sum_{i=1}^{N-1} \pi^{(l)}_i P_{i,j}(t) e^{(j)} + \sum_{i=N}^{N+1} \pi^{(l)}_i P_{i,j} e^{(j)} = \]

\[ = \sum_{i=1}^{N-1} \pi^{(l)}_i \alpha_{i,j} e^{(j)} + \sum_{i=N}^{N+1} \pi^{(l)}_i P_{i,j} e^{(j)}, \]

which does not depend on \( t \). For \( j = N, N+1 \) subblocks \( P_{*,j} \) do not depend on \( t \), thus \( \| \pi^{(l+1)}_j \|_1 = \pi^{(l)}_j P_{*,j} e^{(j)} \) is also independent of \( t \).

b) Now let us suppose that in iteration \( \pi^{(l)} \) subvectors \( N, N+1 \) are independent of \( t \). First, by (4) we have

\[ \pi^{(l+1)}_{N+1} = \sum_{i=1}^{N+1} \pi^{(l)}_i P_{i,N+1} = \sum_{i=N}^{N+1} \pi^{(l)}_i P_{i,N+1}, \]

which by induction hypothesis does not depend on \( t \).

Finally, because of (3),

\[ \pi^{(l+1)}_N = \sum_{i=1}^{N+1} \pi^{(l)}_i P_{i,N} = \sum_{i=1}^{N-1} \pi^{(l)}_i e^{(i)} x^{(i)} T + \pi^{(l)}_N P_{N,N} + \pi^{(l)}_{N+1} P_{N+1,N} = \]

\[ = \sum_{i=1}^{N-1} \| \pi^{(l)}_i \|_1 x^{(i)} T + \pi^{(l)}_N P_{N,N} + \pi^{(l)}_{N+1} P_{N+1,N}, \]

with all terms independent of \( t \).

Remark: The above theorem holds also for periodic chains. If \( P \) is a transition probability matrix of \( p \)-cyclic chain, it has exactly \( p \) eigenvalues on a unit circle (one of them being 1). If we transform matrix \( P \) onto

\[ \tilde{P} = \alpha P + (1 - \alpha) I, \]

111
we obtain a matrix with submatrix of subblocks \((\tilde{P}_{ij}(t))_{i,j=1}^{N-1}\) remaining lumpable and for \(i = 1, \ldots, N - 1\) we will have \(P_{i,N} = e^{(i)} \cdot \alpha x^{(i)T}\). Furthermore \(\pi \tilde{P} = \pi\) and all eigenvalues other than 1 will be inside the unit circle, ensuring convergence of power method.

**Example:** If we change the order of states in Markov chain represented by \(P_c\) to 13, 11, 12, 10, 5, 9, 4, 8, 1, 2, 6, 3, 7, then the resulting chain has a transition probability matrix (zeros omitted)

\[
\tilde{P}_c = \frac{1}{64} \begin{pmatrix}
60 & 60 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 62 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 62 & 62 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 63 & 63 & 62 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 63 & 63 & 62 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 62 & 62 & 62 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 62 & 62 & 62 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 60 & 60 & 60 & 60 & 60 & 60 & 60 & 60 & 60 & 60 & 60
\end{pmatrix},
\]

which is lumpable and if we have perturbations for example
\[
\tilde{p}_{11}(t) = \frac{60}{64} - t, \quad \tilde{p}_{12}(t) = t, \quad \tilde{p}_{66}(t) = \frac{63}{64} - 2t, \quad \tilde{p}_{65}(t) = 2t,
\]
the resulting matrix satisfies conditions of the theorem.

**4 Summary**

This paper intends to present some conditions for insensitivity of a Markov chain towards perturbations in transition probability matrix. These conditions involve existence of cutpoints and regularity described by the concept of lumpability.

**References**


112