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# INSENSITIVITY ANALYSIS OF MARKOV CHAINS\*

Martin Kocurek

## Abstract

Sensitivity analysis of irreducible Markov chains considers an original Markov chain with transition probability matrix  $P$  and modified Markov chain with transition probability matrix  $\tilde{P}$ . For their respective stationary probability vectors  $\pi, \tilde{\pi}$ , some of the following characteristics are usually studied:  $\|\pi - \tilde{\pi}\|_p$  for asymptotical stability [3],  $|\pi_i - \tilde{\pi}_i|, \frac{|\pi_i - \tilde{\pi}_i|}{\pi_i}$  for componentwise stability or sensitivity[1]. For functional transition probabilities,  $P = P(t)$  and stationary probability vector  $\pi(t)$ , derivatives are also used for studying sensitivity of some components of stationary distribution with respect to modifications of  $P$  [2].

In special cases, modifications of matrix  $P$  leave certain stationary probabilities unchanged. This paper studies some special cases which lead to this behavior of stationary probabilities.

## 1 Introduction

A Markov chain is a sequence of random variables  $X_1, X_2, X_3, \dots$ , with the Markov property, namely that, given the present state, the future and past states are independent. Formally,

$$P(X_{n+1} = x | X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = P(X_{n+1} = x | X_n = x_n),$$

where the possible values of  $X_i$  form a countable state space  $S$  of the chain. Markov chains are often described by a directed graph, where the edges are labeled by the probabilities  $p_{ij}$  of moving from state  $i$  to the other state  $j$ . These probabilities are called *transition probabilities* and together they form a *transition probability matrix* denoted by  $P$ , with row sums equal to 1. We will study finite Markov chains (a finite chain has a finite state space  $S = \{x_1, \dots, x_n\}$ ). A state  $i$  has period  $p$  if any return to state  $i$  must occur in multiples of  $p$  time steps. Formally, the period of a state  $i$  is defined as  $p = \gcd\{k : P(X_k = i | X_0 = i) > 0\}$ . If  $p = 1$ , then the state is said to be aperiodic i.e. returns to state  $i$  can occur at irregular times. Otherwise ( $p > 1$ ), the state is said to be periodic with period  $p$ . If all states are periodic with period  $p$ , the chain is called  $p$ -cyclic.

Let us denote

$$e = (1, \dots, 1)^T, \quad e_i = (0, \dots, 0, 1, 0, \dots, 0)^T = (\delta_{i,j})_{j=1}^n, \quad i = 1, \dots, n, \quad P = (P_{ij})_{i,j=1}^n.$$

A Markov chain is called irreducible, if there exists a connection between every two states. That means, matrix  $P$  is irreducible. In this case, matrix  $P$  has a unique

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eigenvalue 1 (which equals to spectral radius  $\rho(P)$  of  $P$ ) and unique left and right eigenvectors associated with this eigenvalue,  $\pi = (\pi_1, \dots, \pi_n)$  and  $e$ , so that

$$\pi P = \pi, \quad P e = e.$$

Vector  $\pi$  is called *stationary probability vector*, we usually normalise this vector to  $\pi e = \|\pi\|_1 = 1$ ;  $i$ -th component  $\pi_i$  of  $\pi$  shows, how often the chain “visits state  $i$ ”,

$$\pi_i = \lim_{m \rightarrow \infty} \frac{|\{j; X_j = x_i, j = 1, \dots, m\}|}{m}.$$

We will also use a different normalisation,  $\pi_k = 1$  and in this case, the eigenvector will be denoted by  $\pi_{(k)}$ , so that  $\pi_{(k)k} = 1$ .

In the following, we will partition matrix  $P$  and vector  $\pi$  into subblocks,

$$\pi = (\pi^{(1)}, \dots, \pi^{(N)}), \quad P = \begin{pmatrix} P_{11} & \dots & P_{1N} \\ \vdots & \ddots & \vdots \\ P_{N1} & \dots & P_{NN} \end{pmatrix}, \quad (1)$$

where  $N$  is the number of subblocks in matrix  $P$ ,  $n_1, \dots, n_N$  will be respective dimensions of subblocks. Conformally with partitioning of  $P$  we shall partition vector  $e = (e^{(1)T}, \dots, e^{(N)T})^T$ , where  $e^{(i)}$  is a vector  $(1, \dots, 1)^T$  with  $n_i$  components.

As an example we will use a Markov chain with the following matrix:

$$P_c = \frac{1}{64} \begin{pmatrix} 62 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 62 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 60 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 62 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 63 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 62 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 60 & 2 & 0 & 0 & 0 & 0 & 0 \\ 62 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 63 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 62 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 62 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 60 \end{pmatrix}. \quad (2)$$

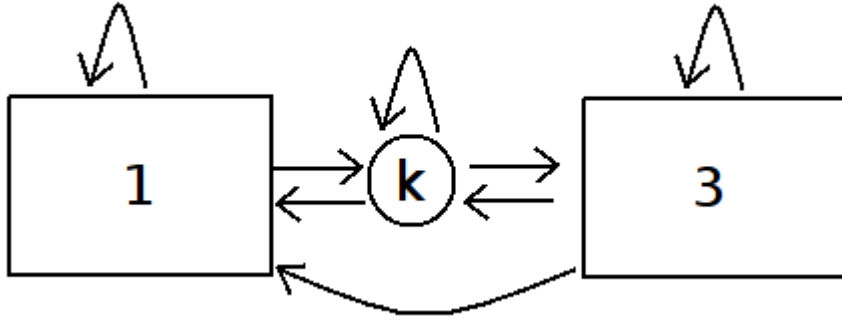
## 2 Normalisation $\pi_k = 1$

Normalisation  $\pi_k = 1$  is useful for computing the eigenvector as a solution of a system of equations  $\pi_{(k)} P = \pi_{(k)}$ , or  $P^T \pi_{(k)}^T = \pi_{(k)}^T$ ,  $(I - P^T) \pi_{(k)}^T = 0$ . By replacing an arbitrary equation with equation  $\pi_{(k)} e_k = 1$ , or equivalently,  $e_k^T \pi_{(k)}^T = 1$ , we obtain a system with better spectral properties, than when using condition  $e^T \pi^T = 1$  [4].

When we use this normalization, we can state the following simple theorem.

**Theorem 1.** Let the state space of a Markov chain can be decomposed into three groups  $S_1, \{x_k\} = S_2, S_3$ , so that in oriented graph of the Markov chain each path from  $S_1$  to  $S_3$  contains a vertex  $x_k$ . Then no modifications of transition probabilities between states of  $S_1$  affect components in  $\pi_{(k)}$  associated with states from  $S_3$

**Proof:** With given restrictions, the graph of the chain can be simplified into



At this picture,  $S_1$  is denoted by 1,  $x_k$  by k,  $S_3$  by 3. It then follows that nonzero structure of  $P$  is

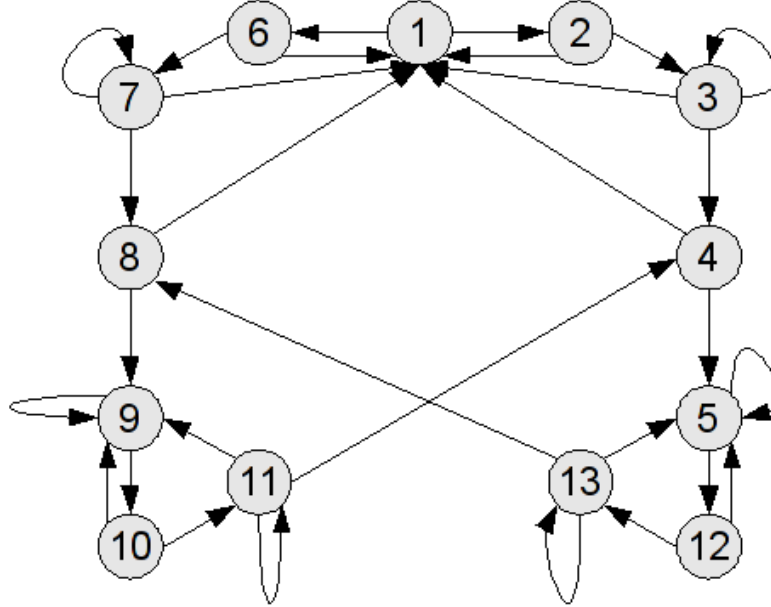
$$P = \left( \begin{array}{ccc|c|ccc} X & \dots & X & X & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X & \dots & X & X & 0 & \dots & 0 \\ \hline X & \dots & X & X & X & \dots & X \\ \hline X & \dots & X & X & X & \dots & X \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X & \dots & X & X & X & \dots & X \end{array} \right).$$

After forming left-hand side matrix  $(I - P^T)$ , we remove  $k$ -th equation and replace it with  $e_k^T \pi_{(k)}^T = 1$ . This way we obtain a system of equations with matrix  $A^{(k)}$  and right-hand side  $e_k$ .  $A^{(k)}$  has the following nonzero structure

$$A^{(k)} = \left( \begin{array}{ccc|c|ccc} X & \dots & X & X & X & \dots & X \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X & \dots & X & X & X & \dots & X \\ \hline 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & X & X & \dots & X \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & X & X & \dots & X \end{array} \right),$$

it is clearly reducible. Thus, no modifications of transition probabilities between states in  $S_1$  (block 1, 1 in  $A^{(k)}$ ) will affect  $k$ -th,  $\dots$ ,  $n$ -th components of  $\pi_{(k)}$

**Example:** In example with  $P_c$ , we may draw an oriented graph:



We see that vertices 2, 3, 6, 7 are accesible only through vertex 1. Thus if we fix the first element of  $\pi$ , no modifications of transition probabilities between vertices 4, 5, 8, 9, 10, 11, 12, 13 will affect components no. 2, 3, 6, 7 in  $\pi_{(1)}$ .

### 3 Normalisation $\pi e = 1$

For a more usual normalisation  $\pi e = 1$ , let us first introduce a concept of lumpability

**Definition:** Let us partition a transition probability matrix  $P$  into blocks  $(P_{ij})_{i,j=1}^N$  so that for every block  $P_{ij}$  and vector  $e^{(j)}$  of appropriate dimensions

$$P_{ij}e^{(j)} = \alpha_{ij}e^{(i)}$$

for some  $\alpha_{ij} \in \mathbf{R}$ . Then matrix  $P$  is said to be lumpable.

**Theorem 2.** Let  $P(t)$  be a perturbed transition probability matrix of an irreducible finite aperiodic Markov chain, whose state space divided into subsets  $S_1, \dots, S_{N+1}$ , where states of  $S_{N+1}$  are accesible only through  $S_N$ . Let perturbations depend on a variable  $t$  and be restricted to lumpable submatrix of blocks  $(P_{ij}(t))_{i,j=1}^{N-1}$ . If for every  $i = 1, \dots, N - 1$  exists a column vector  $x^{(i)}$  such that

$$P_{i,N} = e^{(i)} \cdot x^{(i)T}, \quad (3)$$

then subblocks  $\pi^{(N)}, \pi^{(N+1)}$  are independent of  $t$

**Proof:** From the assumption it follows that

$$P_{ij}e^{(j)} = \alpha_{ij}e^{(j)}, \quad P_{i,N+1} = 0, \quad i, j = 1, \dots, N-1. \quad (4)$$

We will prove the theorem by using a power method for computing  $\pi$ . Assumptions guarantee the existence of a unique steady point – eigenvector  $\pi$  [4].

Let us choose a  $\pi^{(0)} = (\pi_1^{(0)}, \dots, \pi_{N+1}^{(0)})$ , for  $l = 1, 2, \dots$

$$\pi^{(l+1)} = \pi^{(l)}P.$$

a) At first we will show by induction, that for every  $l$  the  $\|\cdot\|_1$ -norms of subvectors  $\pi_1^{(l)}, \dots, \pi_{N+1}^{(l)}$  of  $\pi^{(l)}$  do not depend on  $t$ .  $\pi^{(0)}$  does not depend on  $t$ . The  $l_1$ -norm of the  $j$ -th subvector,  $j = 1, \dots, N-1$ , in the  $(l+1)$ -th iteration is

$$\begin{aligned} \|\pi_j^{(l+1)}\|_1 &= \pi^{(l)}P_{*,j}e^{(j)} = \sum_{i=1}^{N-1} \pi_i^{(l)}P_{i,j}(t)e^{(j)} + \sum_{i=N}^{N+1} \pi_i^{(l)}P_{i,j}e^{(j)} = \\ &= \sum_{i=1}^{N-1} \pi_i^{(l)}\alpha_{i,j}e^{(j)} + \sum_{i=N}^{N+1} \pi_i^{(l)}P_{i,j}e^{(j)}, \end{aligned}$$

which does not depend on  $t$ . For  $j = N, N+1$  subblocks  $P_{*,j}$  do not depend on  $t$ , thus  $\|\pi_j^{(l+1)}\|_1 = \pi^{(l)}P_{*,j}e^{(j)}$  is also independent of  $t$ .

b) Now let us suppose that in iteration  $\pi^{(l)}$  subvectors  $N, N+1$  are independent of  $t$ . First, by (4) we have

$$\pi_{N+1}^{(l+1)} = \sum_{i=1}^{N+1} \pi_i^{(l)}P_{i,N+1} = \sum_{i=N}^{N+1} \pi_i^{(l)}P_{i,N+1},$$

which by induction hypothesis does not depend on  $t$ .

Finally, because of (3),

$$\begin{aligned} \pi_N^{(l+1)} &= \sum_{i=1}^{N+1} \pi_i^{(l)}P_{i,N} = \sum_{i=1}^{N-1} \pi_i^{(l)}e^{(i)}x^{(i)T} + \pi_N^{(l)}P_{N,N} + \pi_{N+1}^{(l)}P_{N+1,N} = \\ &= \sum_{i=1}^{N-1} \|\pi_i^{(l)}\|_1 x^{(i)T} + \pi_N^{(l)}P_{N,N} + \pi_{N+1}^{(l)}P_{N+1,N}, \end{aligned}$$

with all terms independent of  $t$ .

**Remark:** The above theorem holds also for periodic chains. If  $P$  is a transition probability matrix of  $p$ -cyclic chain, it has exactly  $p$  eigenvalues on a unit circle (one of them being 1). If we transform matrix  $P$  onto

$$\tilde{P} = \alpha P + (1 - \alpha)I,$$

