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A NEW RECONSTRUCTION-ENHANCED DISCONTINUOUS GALERKIN METHOD FOR TIME-DEPENDENT PROBLEMS*

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Abstract

This work is concerned with the introduction of a new numerical scheme based on the discontinuous Galerkin (DG) method. We propose to follow the methodology of higher order finite volume schemes and introduce a reconstruction operator into the DG scheme. This operator constructs higher order piecewise polynomial reconstructions from the lower order DG scheme. Such a procedure was proposed already in [2] based on heuristic arguments, however we provide a rigorous derivation, which justifies the increased order of accuracy. Numerical experiments are carried out.

1 Problem formulation and notation

In this paper we shall be concerned with a nonlinear nonstationary scalar hyperbolic equation in a bounded domain $\Omega \subset I\!R^d$ with a Lipschitz-continuous boundary $\partial\Omega$. Let $Q_T := \Omega \times (0, T)$. We treat the following problem:

$$\frac{\partial u}{\partial t} + \mathrm{div}\boldsymbol{f}(\mathrm{u}) = 0 \quad \text{in } \mathrm{Q_T} \tag{1}$$

along with an appropriate initial and boundary condition. Here $\boldsymbol{f} = (f_1, \cdots, f_d)$ and $f_s, s = 1, \ldots, d$ are Lipschitz continuous fluxes in the direction $x_s, s = 1, \ldots, d$.

Let $\mathcal{T}_h$ be a partition (triangulation) of the closure $\overline{\Omega}$ into a finite number of closed simplices $K \in \mathcal{T}_h$. In general we do not require the standard conforming properties of $\mathcal{T}_h$ used in the finite element method (i.e. we admit the so-called hanging nodes). We shall use the following notation. By ∂K we denote the boundary of an element $K \in \mathcal{T}_h$ and set $h_K = \mathrm{diam}(K),\ h = \max_{K\in\mathcal{T}_h} h_K$.

Let $K, K' \in \mathcal{T}_h$. We say that K and K' are *neighbours*, if they share a common *face* $\Gamma \subset \partial K$. By $\mathcal{F}_h$ we denote the system of all faces of all elements $K \in \mathcal{T}_h$. Further, we define the set of all interior and boundary faces, respectively, by

$$\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Omega\}, \qquad \mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}$$

For each $\Gamma \in \mathcal{F}_h$ we define a unit normal vector $\mathbf{n}_\Gamma$, such that for $\Gamma \in \mathcal{F}_h^B$ the normal $\mathbf{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$.

Over a triangulation $\mathcal{T}_h$ we define the *broken Sobolev spaces*

$$H^k(\Omega, \mathcal{T}_h) = \{v;\ v|_K \in H^k(K),\ \forall K \in \mathcal{T}_h\}.$$

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For each face $\Gamma \in \mathcal{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset K_\Gamma^{(L)} \cap K_\Gamma^{(R)}$. We use the convention that $\mathbf{n}_\Gamma$ is the outer normal to $K_\Gamma^{(L)}$. For $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h^I$ we introduce the following notation:

$$v|_\Gamma^{(L)} = \text{ the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \qquad v|_\Gamma^{(R)} = \text{ the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma,$$
$$\langle v \rangle_\Gamma = \tfrac{1}{2}\big(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\big), \qquad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}.$$

On boundary edges $\Gamma \in \mathcal{F}_h^B$, we define $v|_\Gamma^{(R)} = 0$, $[v]_\Gamma = v|_\Gamma^{(L)}$.

Let $n \geq 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$S_h^n = \{v;\, v|_K \in P^n(K), \forall K \in \mathcal{T}_h\},$$

where $P^n(K)$ denotes the space of all polynomials on K of degree $\leq n$.

2 Discontinuous Galerkin (DG) formulation

We multiply (1) by an arbitrary $\varphi_h^n \in S_h^n$, integrate over an element $K \in \mathcal{T}_h$ and apply Green's theorem. By summing over all $K \in \mathcal{T}_h$ and rearranging, we get

$$\frac{d}{dt}\int_\Omega u(t)\,\varphi_h^n\,dx + \sum_{\Gamma \in \mathcal{F}_h}\int_\Gamma \boldsymbol{f}(u)\cdot\mathbf{n}\,[\varphi_h^n]\,dS - \sum_{K \in \mathcal{T}_h}\int_K \boldsymbol{f}(u)\cdot\nabla\varphi_h^n\,dx = 0. \tag{2}$$

The boundary convective terms will be treated similarly as in the finite volume method, i.e. with the aid of a numerical flux $H(u, v, \mathbf{n})$:

$$\int_\Gamma \boldsymbol{f}(u)\cdot\mathbf{n}\,[\varphi_h^n]\,dS \approx \int_\Gamma H(u^{(L)}, u^{(R)}, \mathbf{n})[\varphi_h^n]\,dS. \tag{3}$$

We assume that H is *Lipschitz continuous, consistent* and *conservative*, cf. [4].

Thus, we obtain the following standard DG formulation

$$\frac{d}{dt}\big(u_h(t), \varphi_h^n\big) + b_h\big(u_h(t), \varphi_h^n\big) = 0, \quad \forall \varphi_h^n \in S_h^n,\ \forall t \in (0, T), \tag{4}$$

where $b_h(\cdot,\cdot)$ is the *convective form* defined for $v, \varphi \in H^1(\Omega, \mathcal{T}_h)$:

$$b_h(v, \varphi) = \int_{\mathcal{F}_h} H(v^{(L)}, v^{(R)}, \mathbf{n})[\varphi]\,dS - \sum_{K \in \mathcal{T}_h}\int_K \boldsymbol{f}(v)\cdot\nabla\varphi\,dx.$$

3 Reconstructed discontinuous Galerkin (RDG) formulation

For $v \in L^2(\Omega)$, we denote by $\Pi_h^n v$ the $L^2(\Omega)$-projection of v on S_h^n:

$$\Pi_h^n v \in S_h^n, \quad (\Pi_h^n v - v,\, \varphi_h^n) = 0, \qquad \forall\, \varphi_h^n \in S_h^n. \tag{5}$$

The basis of the proposed method lies in the observation that (2) can be viewed as an equation for the evolution of $\Pi_h^n u(t)$, where u is the exact solution of (1). In other words, due to (5), $\Pi_h^n u(t) \in S_h^n$ satisfies the following equation for all $\varphi_h^n \in S_h^n$:

$$\frac{d}{dt}\int_\Omega \Pi_h^n u(t)\,\varphi_h^n\,dx + \int_{\mathcal{F}_h} \boldsymbol{f}(u)\cdot\mathbf{n}\,[\varphi_h^n]\,dS - \sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{f}(u)\cdot\nabla\varphi_h^n\,dx = 0. \tag{6}$$

Now, let $N > n$ be an integer. We assume, that there exists a piecewise polynomial function $U_h^N(t) \in S_h^N$, which is an approximation of $u(t)$ of order $N+1$, i.e.

$$U_h^N(x,t) = u(x,t) + O(h^{N+1}), \quad \forall x\in\Omega,\ \forall t\in[0,T]. \tag{7}$$

This is possible, if u is sufficiently regular in space, e.g. $u(t) \in W^{N+1,\infty}(\Omega)$, cf.[1]. Now we incorporate the approximation $U_h^N(t)$ into (6): the exact solution u satisfies

$$\frac{d}{dt}\big(\Pi_h^n u(t),\varphi_h^n\big) + b_h\big(U_h^N(t),\varphi_h^n\big) = E(\varphi_h^n), \quad \forall\varphi_h^n\in S_h^n,\ \forall t\in(0,T), \tag{8}$$

where $E(\varphi_h^n)$ is an error term defined as

$$E(\varphi_h^n) = b_h\big(U_h^N(t),\varphi_h^n\big) - b_h\big(u(t),\varphi_h^n\big). \tag{9}$$

Lemma 31 *The following estimate holds:*

$$E(\varphi_h^n) = O(h^N)\|\varphi_h^n\|_{L^2(\Omega)}. \tag{10}$$

Proof: Due to assumptions (H1) and (H2) it is easy to see that on an edge $\Gamma\in\mathcal{F}_h$

$$\boldsymbol{f}(u)\cdot\mathbf{n} - H(U_h^{N,(L)},U_h^{N,(R)},\mathbf{n}) = H(u,u,\mathbf{n}) - H(U_h^{N,(L)},U_h^{N,(R)},\mathbf{n}) = O(h^{N+1}).$$

Furthermore, due to the Lipschitz-continuity of f_s, we have on element $K\in\mathcal{T}_h$

$$\boldsymbol{f}(u) - \boldsymbol{f}(U_h^N) = O(h^{N+1}).$$

Estimate (10) follows from these results and the application of the *inverse* and *multiplicative trace inequalities*, cf [4]. □

It remains to construct a sufficiently accurate approximation $U_h^N(t)\in S_h^N$ to $u(t)$, such that (7) is satisfied. This leads to the following problem.

Definition 32 (Reconstruction problem.) *Let $v:\Omega\to\mathbb{R}$ be sufficiently regular. Given $\Pi_h^n v\in S_h^n$, find $v_h^N\in S_h^N$ such that $v - v_h^N = O(h^{N+1})$ in Ω. We define the corresponding reconstruction operator $R: S_h^n\to S_h^N$ by $R\,\Pi_h^n v := v_h^N$.*

By setting $U_h^N(t) := R\,\Pi_h^n u(t)$ in (8)-(10), we obtain the following equation for the $L^2(\Omega)$-projections of the exact solution u onto the space S_h^n:

$$\frac{d}{dt}\big(\Pi_h^n u(t),\varphi_h^n\big) + b_h\big(R\,\Pi_h^n u(t),\varphi_h^n\big) = O(h^N)\|\varphi_h^n\|_{L^2(\Omega)}, \quad \forall\varphi_h^n\in S_h^n. \tag{11}$$

By neglecting the right-hand side term and introducing the approximation $u_h^n(t) \approx \Pi_h^n u(t)$, we arrive at the following definition of the *reconstructed discontinuous Galerkin* (RDG) scheme. We seek u_h^n such that

$$\frac{d}{dt}\big(u_h^n(t), \varphi_h^n\big) + b_h\big(Ru_h^n(t), \varphi_h^n\big) = 0, \quad \forall \varphi_h^n \in S_h^n,\ \forall t \in (0,T). \tag{12}$$

There are several points worth mentioning.

- The derivation of the RDG scheme follows the methodology of higher order finite volume schemes and spectral volume schemes, cf. [7]. The basis of these schemes is an equation for the evolution of averages of the exact solution on individual elements (i.e. an equation for $\Pi_h^0 u(t)$). Equation (11) is a direct generalization for the case of higher order $L^2(\Omega)$-projections $\Pi_h^n u(t)$, $n \geq 0$.
- Both $u_h^n(t)$ and φ_h^n lie in S_h^n. Only $Ru_h^n(t)$, lies in the higher dimensional space S_h^N. Despite this fact, equation (11) indicates, that we may expect $u - Ru_h^n = O(h^{N+1})$, although $u - u_h^n = O(h^{n+1})$.
- Numerical quadrature must be employed to evaluate surface and volume integrals in (12). Since test functions are in S_h^n, as compared to S_h^N in the corresponding Nth order standard DG scheme, we may use lower order (i.e. more efficient) quadrature formulae as compared to standard DG.
- In practice, an explicit time discretization must be applied to (12) The upper limit on stable time steps, given by a CFL-like condition, is more restrictive with growing N. However, in the RDG scheme, stability properties are inherited from the lower order scheme, therefore a larger time step is possible as compared to the standard DG scheme.

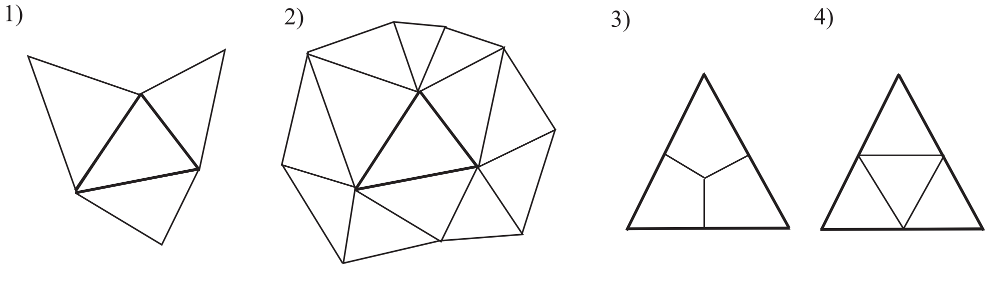

Fig. 1: *1) FV stencil for linear reconstruction, 2) FV stencil for quadratic reconstruction, 3) Control volumes in a spectral volume for linear reconstruction, 4) Analogy to the SV approach for DG - partition of triangle into control volumes, e.g. cubic reconstruction from linear data.*

3.1 Construction of the reconstruction operator

In analogy to the construction of reconstruction operators in higher order FV schemes, we propose two approaches.

3.1.1 'Standard' approach

In the *standard approach*, a stencil (a group of neighboring elements and the element under consideration) is used to build an Nth-degree polynomial approximation to u on the element under consideration ([5] [6]). In the FV method, the von Neumann neighborhood of an element is used as a stencil to obtain a piecewise linear reconstruction, Figure 1, 1). However, for higher order reconstructions, the size of the stencil increases dramatically, Figure 1, 2), rendering higher degrees than quadratic very time consuming. In the case of the RDG scheme, we need not increase the stencil size to obtain higher order accuracy, it suffices to increase the order of the underlying DG scheme.

The reconstruction operator R is constructed analogously as in the FV method, so that $R\Pi_h^n$ is in some sense *polynomial preserving*. Specifically, for each element K and its corresponding stencil S, we require that for all $p \in P^N(S)$

$$\Big((R\Pi_h^n)\big|_S\, p \Big)\Big|_K = p\big|_K. \tag{13}$$

This requirement allows us to study approximation properties of R using the Bramble-Hilbert technique as in the standard finite element method, [1]. The disadvantage of this approach is that for unstructured meshes, the coefficients of the reconstruction operator must be stored for each individual stencil.

3.1.2 Spectral volume approach

In the *spectral volume approach*, we start with a partition of Ω into so-called *spectral volumes* S, for example triangles in 2D. The triangulation $\mathcal{T}_h$ is formed by subdividing each spectral volume S into sub-cells K, called *control volumes*, [7]. In the FV method, the order of accuracy of the reconstruction determines the number of control volumes to be generated in each spectral volume. For example, for a linear reconstruction on a triangle, the triangle is divided into three control volumes, Figure 1, 3). Again, in the RDG scheme, we may use only the smallest available partition into control volumes, and increase the accuracy by increasing the order of the underlying scheme, cf. Figure 1, 4). The reconstruction operator should again be polynomial preserving, i.e. constructed similarly to (13).

The advantage of this approach is that, since all spectral volumes are affine equivalent, it is sufficient to construct the reconstruction operator R only on a reference spectral volume.

4 Numerical experiments

We present preliminary numerical experiments for the periodic advection of a 1D sine wave on uniform meshes. Experimental orders of accuracy α in various norms on meshes with N elements are given in Tables 1 and 2. The increase in accuracy due to reconstruction is clearly visible.

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathcal{T}_h)}$	α
4	5.82E-03	–	3.49E-03	–	3.65E-02	–
8	7.53E-05	6.27	4.43E-05	6,30	1.06E-03	5,11
16	9.07E-07	6.38	5.95E-07	6,22	3.58E-05	4,89
32	1.82E-08	5.64	8.70E-09	6,10	1.16E-06	4,95
64	3.41E-10	5.74	1.33E-10	6,03	3.67E-08	4,98

Tab. 1: *1D advection of sine wave, P^1 RDG scheme with P^5 reconstruction.*

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathcal{T}_h)}$	α
4	2.90E-03	–	1.85E-03	–	1.63E-02	–
8	7.75E-06	8.55	3.56E-06	9.02	1.03E-04	7.30
16	2.10E-08	8.53	6.64E-09	9.07	4.34E-07	7.89
32	7.21E-11	8.18	4.02E-11	7.37	1.76E-09	7.94

Tab. 2: *1D advection of sine wave, P^2 RDG scheme with P^8 reconstruction.*

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